Set-membership identification of block-structured nonlinear feedback systems

V. Cerone, D. Piga, D. Regruto
Dipartimento di Automatica e Informatica
Politecnico di Torino (Italy)

e-mail: vito.cerone@polito.it, dario.piga@polito.it, diego.regruto@polito.it
Nonlinear feedback systems

\[ \mathcal{N} \text{: nonlinear static block} \]
\[ \mathcal{L} \text{: linear dynamic subsystem} \]
\[ x_t, \nu_y: \text{not measurable inner signals} \]
\[ u_t: \text{known input signal} \]
\[ y_t: \text{noise-corrupted measurement of } w_t \]

\[ \nu_t = \mathcal{N}(w_t) = \sum_{k=1}^{n} \gamma_k w_t^k \text{ with } n: \text{polynomial degree} \]

\[ w_t = \frac{B(q^{-1})}{A(q^{-1})} x_t \text{ with } \]
\[ A(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_n q^{-na} \]
\[ B(q^{-1}) = b_0 + b_1 q^{-1} + \ldots + b_{nb} q^{-nb} \]
\[ q^{-1} w_t = w_{t-1} \]
Problem formulation

- **Aim:** compute bounds on the parameters $\gamma^T = [\gamma_1, \gamma_2 \ldots \gamma_n]$ and $\theta^T = [a_1 \ldots a_{na} b_0 \ldots b_{nb}]$

- **Prior assumption on the system:**
  - BIBO stability
  - $na$ and $nb$ are known
  - $n$ is finite and known
  - the steady-state gain of the linear subsystem is not zero
  - a rough upper bound on the settling time of the system is known

- **Prior assumption on the measurement uncertainty:**
  - $\eta_t$ is UBB: $|\eta_t| \leq \Delta \eta_t$
  - $\Delta \eta_t$ is known
Proposed solution

Three-stage procedure:

- **First stage**: computation of bounds on the nonlinear block parameters $\gamma$.

- **Second stage**: computation of bounds on the inner (unmeasurable) signal $x_t$.

- **Third stage**: computation of bounds on the linear block parameters $\theta$. 
Proposed solution: first stage

Bounds on the parameters $\gamma$ of the nonlinear block:

- Stimulate the system with square-wave of $M$ different amplitude and get steady-state measurements
- The feasible parameters set $\mathcal{D}_\gamma$ of the nonlinear block is described as:
\[
\mathcal{D}_\gamma = \{ \gamma \in \mathbb{R}^n : (\bar{y}_s - \bar{\eta}_s) + \sum_{k=1}^{n} \gamma_k (\bar{y}_s - \bar{\eta}_s)^k = \bar{u}_s, \\
| \bar{\eta}_s | \leq \Delta \bar{\eta}_s; \quad s = 1, \ldots, M \},
\]
- $\mathcal{D}_\gamma$ is the set of all parameters $\gamma$ consistent with the $M$ given measurements, the error bounds and the assumed model structure
- Bounds on parameter $\gamma_k$:
\[
\gamma_k^{\min} = \min_{\gamma \in \mathcal{D}_\gamma} \gamma_k \quad \quad \gamma_k^{\max} = \max_{\gamma \in \mathcal{D}_\gamma} \gamma_k
\]
Proposed solution: first stage

Computation of $\gamma_k^{\text{min}}$ and $\gamma_k^{\text{max}}$:

$$\gamma_k^{\text{min}} = \min_{(\gamma, \eta) \in \mathcal{D}_{\gamma \eta}} \gamma_k \quad \gamma_k^{\text{max}} = \max_{(\gamma, \eta) \in \mathcal{D}_{\gamma \eta}} \gamma_k$$

Here:

$$\eta = [\eta_1 \ \eta_2 \ \ldots \ \eta_M]^T,$$

$$\mathcal{D}_{\gamma \eta} = \{ (\gamma, \eta) \in \mathbb{R}^n \times \mathbb{R}^M : (\bar{y}_s - \bar{\eta}_s) + \sum_{k=1}^{n} \gamma_k (\bar{y}_s - \bar{\eta}_s)^k = \bar{u}_s, \quad |\eta_s| \leq \Delta \bar{\eta}_s; \ s = 1, \ldots, M \}$$

$\mathcal{D}_{\gamma \eta}$ is a semialgebraic set over $\mathbb{R}^{n+M}$

The above problems are semialgebraic (nonconvex) optimization problems
Proposed solution: first stage

Standard **nonlinear optimization tools** can not be exploited to compute $\gamma^\text{min}_k$ and $\gamma^\text{max}_k$ since they can trap in **local minima**

$$\Downarrow$$

The true value of $\gamma_k$ could not lie in $[\gamma^\text{min}_k, \gamma^\text{max}_k]$

**Relax** original identification problems to **convex** optimization problems

$$\Downarrow$$

**Bounds** on each parameter $\gamma_k$ can be obtained
Convex relaxation

LMI relaxation for semialgebraic optimization problems:
- SOS decomposition
- Theory of moments

$k$-relaxed bounds $\gamma_k^{min} \delta$ and $\gamma_k^{max} \delta$ computed solving the following SDP problems:

$$
\gamma_k^{min} \delta = \min_{x \in D^\delta_x} f(x) \\
\gamma_k^{max} \delta = \max_{x \in D^\delta_x} f(x)
$$

Here:
- LMI decision variables
- $x$) : linear function
- $D^\delta_x$ : Convex set described by LMI constraints
Tightness and convergence

Property 1 — $\delta$-relaxed bounds become tighter as $\delta$ increases:

$$
\gamma_{min}^\delta_k \leq \gamma_{min}^{\delta+1}_k \leq \gamma_{min}^k
$$

$$
\gamma_{max}^\delta_k \geq \gamma_{max}^{\delta+1}_k \geq \gamma_{max}^k
$$

Property 2 — $\delta$-relaxed bounds converge to the true bounds as $\delta \to \infty$:

$$
\lim_{\delta \to \infty} \gamma_{min}^\delta_k = \gamma_{min}^k
$$

$$
\lim_{\delta \to \infty} \gamma_{max}^\delta_k = \gamma_{max}^k
$$
Computational complexity of the LMI relaxation

In practice, due to an high computational complexity, LMI relaxation techniques can be exploited only for a small set of measurements

↓

A reduction of the complexity of SDP relaxed problems is necessary
Reduced complexity of the relaxed problems

\[ D_{\gamma\eta} = \left\{ \left( \gamma, \eta \right) \in \mathbb{R}^n \times \mathbb{R}^M : \left( \bar{y}_s - \bar{\eta}_s \right) + \sum_{k=1}^{n} \gamma_k \left( \bar{y}_s - \bar{\eta}_s \right)_k = \bar{u}_s, \right\} \]

\[ \left| \bar{\eta}_s \right| \leq \Delta \bar{\eta}_s; \quad s = 1, \ldots, M \}

**Property 3** The variables \( \bar{\eta}_s \) defining \( D_{\gamma\eta} \) are not correlated with each other.

**Note** In constructing moment matrix defining \( D_{\delta x} \) do not consider the correlation between variables \( \bar{\eta}_s \).
Reduced complexity of the relaxed problems

Value of $M$ greater than 400 can be exploited in the identification (for $\delta \leq 4$)

Property 4 — Convergence to tight bounds is preserved
Proposed solution: second stage

Bounds on the inner signal $x_t$:

$$
x_t^{min} = u_t - \nu_t^{max}
$$

$$
x_t^{max} = u_t - \nu_t^{min}
$$

$$
\nu_t^{min} = \min_{(\gamma, \eta) \in D_{\gamma, \eta}, |\eta| \leq \Delta \eta} \sum_{k=1}^{n} \gamma_k (y_t - \eta)^k
$$

$$
\nu_t^{max} = \max_{(\gamma, \eta) \in D_{\gamma, \eta}, |\eta| \leq \Delta \eta} \sum_{k=1}^{n} \gamma_k (y_t - \eta)^k
$$

- Stimulate the system with a persistently exciting input signal $u_t$
- Bounds on $\nu_t$ can be computed by means of LMI relaxation
- Structure of the problem can be exploited to reduce the computation complexity
Proposed solution: third stage

Bounds on the linear block parameters $\theta$:

Inner signal $x_t$ described in terms of its central value $x^c_t$ and its perturbation $\delta x_t$:

$$x_t = x^c_t + \delta x_t$$

such that:

$$|x_t| \leq \Delta x_t, \quad x^c_t = \frac{x_{t}^{\min} + x_{t}^{\max}}{2}, \quad \Delta x_t = \frac{x_{t}^{\max} - x_{t}^{\min}}{2}$$

Identification of a linear model with noisy output sequence $\{y_t\}$ and uncertain input sequence $\{x_t\}$

Errors-in-variables (EIV) problem with bounded errors
Proposed solution: bounds on $\theta$

Exploiting previous results on EIV problems with bounded errors.

Cerone, “Feasible parameter set of linear models with bounded errors in all variables”, *Automatica* 1993

$$\downarrow$$

Bounds on $\theta_j$ are computed by means of linear programming.
Example

Parameters of the simulated system:

\[ (w_t) = -1.5w_t + 1.2w_t^2 + 0.9w_t^3 \]

\[ (q^{-1}) = 1 - 1.5193q^{-1} + 0.5326q^{-2} \]

\[ (q^{-1}) = 0.1549q^{-1} - 0.1416q^{-2} \]

Measurements output errors:

\[ s \mid \leq \Delta \bar{\eta}_s, \quad \{ \bar{\eta}_s \} \text{ random variables belonging to } [-\Delta \bar{\eta}_s, +\Delta \bar{\eta}_s] \]

\[ t \mid \leq \Delta \eta_t, \quad \{ \eta_t \} \text{ random variables belonging to } [-\Delta \eta_t, +\Delta \eta_t] \]

During the simulated experiment the SNR is about 25db.
Nonlinear block parameters: central estimates and parameters bounds ($M = 50, \delta = 3$)

<table>
<thead>
<tr>
<th>True Value</th>
<th>$\gamma_k^{min}$</th>
<th>$\gamma_k^c$</th>
<th>$\gamma_k^{max}$</th>
<th>$\Delta \gamma_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.5000</td>
<td>-1.5369</td>
<td>-1.4890</td>
<td>1.4410</td>
<td>0.0480</td>
</tr>
<tr>
<td>1.2000</td>
<td>1.1931</td>
<td>1.2072</td>
<td>1.2213</td>
<td>0.0141</td>
</tr>
<tr>
<td>0.9000</td>
<td>0.8898</td>
<td>0.9020</td>
<td>0.9141</td>
<td>0.0121</td>
</tr>
</tbody>
</table>

$$\Delta \gamma_k = \frac{\gamma_k^{max} - \gamma_k^{min}}{2}$$
## Linear block parameters: central estimates and parameters bounds

<table>
<thead>
<tr>
<th>N</th>
<th>True Value</th>
<th>$\theta_j^{\text{min}}$</th>
<th>$\theta_j^c$</th>
<th>$\theta_j^{\text{max}}$</th>
<th>$\Delta \theta_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>-1.5193</td>
<td>-2.0326</td>
<td>-1.6422</td>
<td>-1.2518</td>
<td>0.3904</td>
</tr>
<tr>
<td></td>
<td>0.5326</td>
<td>0.3046</td>
<td>0.6364</td>
<td>0.9681</td>
<td>0.3318</td>
</tr>
<tr>
<td></td>
<td>0.1549</td>
<td>0.1424</td>
<td>0.1579</td>
<td>0.1734</td>
<td>0.0155</td>
</tr>
<tr>
<td></td>
<td>-0.1416</td>
<td>-0.2201</td>
<td>-0.1232</td>
<td>-0.0264</td>
<td>0.0969</td>
</tr>
<tr>
<td>300</td>
<td>-1.5193</td>
<td>-1.8569</td>
<td>-1.5633</td>
<td>-1.2697</td>
<td>0.2936</td>
</tr>
<tr>
<td></td>
<td>0.5326</td>
<td>0.3265</td>
<td>0.5761</td>
<td>0.8256</td>
<td>0.2496</td>
</tr>
<tr>
<td></td>
<td>0.1549</td>
<td>0.1452</td>
<td>0.1555</td>
<td>0.1659</td>
<td>0.0104</td>
</tr>
<tr>
<td></td>
<td>-0.1416</td>
<td>-0.1951</td>
<td>-0.1348</td>
<td>-0.0746</td>
<td>0.0602</td>
</tr>
</tbody>
</table>

$$\Delta \theta_j = \frac{\theta_j^{\text{max}} - \theta_j^{\text{min}}}{2}$$
Conclusion

- Three stage procedure to evaluate parameters bounds of a nonlinear feedback system
- Bounds on the nonlinear block parameters have been evaluated by means of LMI relaxation techniques
- The particular structure of the identification problems allows the reduction of the complexity of the LMI relaxation
- Convergence to tight bounds is guaranteed
- Bounds on the parameters of the linear block has been computed through the evaluation of bounds on the unmeasurable inner signal $x_t$