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Improved parameter bounds for set-membership EIV problems

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SUMMARY

In this paper, we consider the set-membership error-in-variables identification problem, that is the identification of linear dynamic systems when output and input measurements are corrupted by bounded noise. A new approach for the computation of parameters uncertainty intervals is presented. First, the problem is formulated in terms of nonconvex optimization. Then, a relaxation procedure is proposed to compute parameter bounds by means of semidefinite programming techniques. Finally, accuracy of the estimate and computational complexity of the proposed algorithm are discussed. Advantages of the proposed technique with respect to previously published ones are discussed both theoretically and by means of a simulated example.

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KEY WORDS: set-membership identification; errors-in-variables; LMI relaxation

1. INTRODUCTION

System identification addresses the problem of deriving the mathematical model of dynamic systems on the basis of three basic ingredients: a set of input–output measurements, a set of candidate models and an identification method (see, e.g. [1]). Since in real-world applications, data sequences are experimentally collected, both input and output samples are corrupted by measurement noise. Identification problems where both input and output measurements are affected by noise are referred to as errors-in-variables (EIV) problems. Although a good deal of techniques have been proposed in the last decades to tackle the problem, its challenging features are continuously stimulating new research effort in the system identification community. Most of the proposed contributions rely on a stochastic description of measurement errors. Instrumental variable approaches are applied to systems described both in input–output [2, 3] and in state-space [4] form. Many researchers have contributed to the development of the bias compensated least-squares approach (see., e.g. the paper by Zheng [5] for a recent work on the subject). Consistency of total least-squares-based methods is analyzed by Kukush and co-workers in [6], whereas frequency domain approaches are considered in [7]. The first attempt to solve the problem through a suitable extension of the Frisch scheme is presented in [8] by Beghelli and co-workers. Further refinements of this approach can be found, for instance, in the work by Soderstrom *et al.* [9]. Results on maximum likelihood identification for EIV problems are reported by Pintelon and Schoukens [10] and by Diversi *et al.* [11]. A detailed review of the main contributions on identification of linear dynamic systems, when input and output measurements are corrupted by stochastic noise, can be found in the survey paper by Soderstrom [12].

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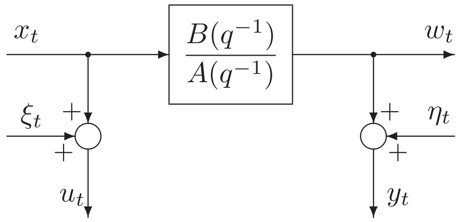

Figure 1. Errors-in-variables basic setup for linear dynamic system.

A possible alternative to the stochastic description of measurement noise is the bounded error characterization, which leads to the set-membership identification framework. Details on this approach can be found in a number of survey papers (see, e.g. [13, 14]), in the book edited by Milanese *et al.* [15], and the special issues edited by Norton [16, 17]. To the best of the authors' knowledge, only few contributions on EIV identification with bounded noise are reported in the literature. First attempts to address the problem can be found in papers [18, 19] where ARMAX models with bounded errors are considered. A detailed analysis of this problem is presented by Veres and Norton in [19] where they show that the exact feasible parameter region of dynamic EIV models is described by nonlinear bounds, whose shape may become fairly complex when the number of data increases. As a consequence, parameter bounds cannot be easily computed and the use of either polytopic or ellipsoidal outer approximation is suggested. For the case of static models, an exact description of the feasible parameter set (FPS) is provided in paper [20] where topological features, such as convexity and connectedness, are also discussed. In [21], results from [20] are applied to the problem of EIV identification of linear dynamic systems. In that paper, an outer approximation of the true nonconvex FPS is obtained as the union of a finite number of polytopes. This outer-bounding set is then used to compute the parameter uncertainty intervals (PUIs) through the solution of linear programming (LP) problems. The resulting PUIs are not tight and the degree of conservativeness of the approach proposed in [21] is, in general, not easy to quantify.

In this paper we present an alternative approach, based on linear matrix inequalities (LMI) relaxation techniques, for the computation of parameter bounds of EIV dynamic models less conservative than those obtained in [21]. The note is organized as follows. Section 2 is devoted to the formulation of the problem. The relaxation procedure presented in [21] is briefly reviewed in Section 3 for completeness and self-consistency of the paper. Then, the new relaxation approach is presented in Section 4, whereas its properties are analyzed in detail in Section 5. A simulated example is reported in Section 6 in order to show the effectiveness of the proposed technique. Concluding remarks end the paper.

2. PROBLEM FORMULATION

Consider the single-input single-output (SISO) linear-time-invariant (LTI) system depicted in Figure 1, where x_t is the noise-free input sequence and the linear block is modeled by a discrete time system that transforms x_t into the noise-free output w_t according to the difference equation

$$A(q^{-1})w_t = B(q^{-1})x_t, \tag{1}$$

where $A(\cdot)$ and $B(\cdot)$ are polynomials in the backward shift operator q^{-1} ($q^{-1}w_t = w_{t-1}$) of the form

$$A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_{na} q^{-na}, \tag{2}$$

$$B(q^{-1}) = b_0 + b_1 q^{-1} + \cdots + b_{nb} q^{-nb}. \tag{3}$$

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Both input and output data sequences are corrupted by additive noise ξ_t and η_t, respectively,

$$u_t = x_t + \xi_t, \tag{4}$$

$$y_t = w_t + \eta_t, \tag{5}$$

where ξ_t and η_t are assumed to range within given bounds $\Delta\xi_t$ and $\Delta\eta_t$, respectively, that is

$$|\xi_t| \leqslant \Delta\xi_t, \tag{6}$$

$$|\eta_t| \leqslant \Delta\eta_t. \tag{7}$$

The unknown parameter vector $\theta \in \mathbb{R}^p$ to be identified is defined as

$$\theta = [a_1 \ \ldots \ a_{na} \ b_0 \ b_1 \ \ldots \ b_{nb}]^{\mathrm{T}}, \tag{8}$$

where $na + nb + 1 = p$, whereas the FPS $\mathscr{D}_\theta$ is

$$\begin{aligned} \mathscr{D}_\theta = \{&\theta \in \mathbb{R}^p : A(q^{-1})(y_t - \eta_t) = B(q^{-1})(u_t - \xi_t), \\ &|\xi_t| \leqslant \Delta\xi_t, \ |\eta_t| \leqslant \Delta\eta_t; \ t = 1, \ldots, N\}, \end{aligned} \tag{9}$$

where N is the length of data sequences. Equation (9) provides an exact description of the set of all possible values of the unknown parameter θ consistent with measured data, error bounds and assumed model structure. It is worth noting that $\mathscr{D}_\theta$ is the projection over the parameters space $\mathbb{R}^p$ of the set $\mathscr{D}_{\theta\xi\eta}$, defined as

$$\begin{aligned} \mathscr{D}_{\theta\xi\eta} &= \{(\theta, \xi, \eta) \in \mathbb{R}^p \times \mathbb{R}^N \times \mathbb{R}^N : A(q^{-1})(y_t - \eta_t) = B(q^{-1})(u_t - \xi_t), \\ &\quad |\xi_t| \leqslant \Delta\xi_t, \ |\eta_t| \leqslant \Delta\eta_t; \ t = 1, \ldots, N\}, \\ &= \{(\theta, \xi, \eta) \in \mathbb{R}^p \times \mathbb{R}^N \times \mathbb{R}^N : \sum_{i=0}^{na} a_i (y_{t-i} - \eta_{t-i}) = \sum_{j=0}^{nb} b_j (u_{t-j} - \xi_{t-j}), \\ &\quad |\xi_t| \leqslant \Delta\xi_t, |\eta_t| \leqslant \Delta\eta_t; \ t = 1, \ldots, N\}, \end{aligned} \tag{10}$$

where $\xi = [\xi_1, \xi_2, \ldots, \xi_N]^{\mathrm{T}} \in \mathbb{R}^N$ and $\eta = [\eta_1, \eta_2, \ldots, \eta_N]^{\mathrm{T}} \in \mathbb{R}^N$. In this work, we address the problem of evaluating the PUI_j, defined as:

$$\mathrm{PUI}_j = [\underline{\theta}_j, \ \overline{\theta}_j] \quad \text{for } j = 1, \ldots, p, \tag{11}$$

where

$$\underline{\theta}_j = \min_{\theta, \xi, \eta \in \mathscr{D}_{\theta\xi\eta}} \theta_j, \tag{12}$$

$$\overline{\theta}_j = \max_{\theta, \xi, \eta \in \mathscr{D}_{\theta\xi\eta}} \theta_j. \tag{13}$$

Thus, the computation of the PUI_j requires the solution to constrained optimization problems (12) and (13). Since $\mathscr{D}_{\theta\xi\eta}$ is a nonconvex set (see, e.g. [19]), standard nonlinear optimization tools (gradient method, Newton method, etc.) cannot be used because they can trap in local minima. As a consequence, the PUI_js obtained using these tools are not guaranteed to contain the true unknown parameter, which is a key requirement of any set-membership identification method. A possible solution to overcome this problem is to relax (12) and (13) to convex problems in order to obtain a lower (upper) bound of $\underline{\theta}_j$ ($\overline{\theta}_j$).

It must be pointed out that (12) and (13) are polynomial-constrained optimization problems since the objective function is linear and the feasible region $\mathscr{D}_{\theta\xi\eta}$ is a semialgebraic set over the space $\mathbb{R}^p \times \mathbb{R}^N \times \mathbb{R}^N$ (see [22] for details). Remarkable efforts have been devoted to relax constrained polynomial problems by a hierarchy of convex LMI optimization problems (see survey paper [23] for a literature review on the subject). The approach proposed by Parrillo in [24] is based on the representation of nonnegative polynomials as sum of squares (SOS), whereas Lasserre in [25] exploits the dual theory of moments. More specifically, the relaxation technique described in [25] solves semidefinite optimization problems, whose solutions are guaranteed to converge monotonically to the global optima of the semialgebraic optimization problem when the length of the number of successive LMI relaxations, the relaxation order δ, increases. The use of these LMI-relaxation techniques, due to their high computational complexity, is restricted to the relaxation of polynomial optimization problems with a small number of optimization variables, which in general are not greater than 20 for relaxation order smaller than 4. Because the number of the optimization variables in problems (12) and (13) is $p+2N$, LMI-relaxation techniques can be used in practice to relax (12) and (13) only if the measurement number N is small (in general not greater than 10), unusual and unfavorable condition in the identification framework. A possible way to overcome this restriction is to exploit the approach presented in [26, 27], where it is shown that the LMI-relaxation computational complexity can be significantly reduced when the problem exhibits a sparsity structure. Application of results from [26, 27] to the set-membership EIV problem is investigated in [22]. In this paper, we present an alternative approach to relax (12) and (13) to convex optimization problems. The proposed method involves only p optimization variables, thus allowing numerical computation of parameter bounds also for large value of N. Furthermore, the new relaxation procedure provides parameter uncertainty intervals tighter than the ones obtained in [21], where the static EIV method of [20] is exploited.

3. STATIC EIV RELAXATION

The main idea in [21] is to relax problems (12)–(13) assuming that uncertain regressors are uncorrelated. In this way, an outer approximation $\mathcal{D}_\theta^s$ of $\mathcal{D}_\theta$ can be obtained, whose formal description is given by the following result.

Result 1—Description of the outer-bound $\mathcal{D}_\theta^s$ [21]

The set $\mathcal{D}_\theta^s$ obtained from $\mathcal{D}_\theta$, assuming that the uncertain regressor is uncorrelated, has the form

$$\mathcal{D}_\theta^s = \{\theta \in \mathbb{R}^p : (\phi_t - \Delta\phi_t)\theta \leqslant y_t + \Delta\eta_t,\\ (\phi_t + \Delta\phi_t)\theta \geqslant y_t - \Delta\eta_t;\ t = 1, \ldots, N\}, \tag{14}$$

where

$$\Delta\phi_t = [\Delta\eta_{t-1}\,\mathrm{sgn}(a_1)\ldots\Delta\eta_{t-na}\,\mathrm{sgn}(a_{na})\Delta\xi_t\,\mathrm{sgn}(b_0)\Delta\xi_{t-1}\,\mathrm{sgn}(b_1)\ldots\Delta\xi_{t-nb}\,\mathrm{sgn}(b_{nb})]^{\mathrm{T}}. \tag{15}$$

The set $\mathcal{D}_\theta^s$ is an outer approximation of $\mathcal{D}_\theta$, i.e. $\mathcal{D}_\theta^s \supseteq \mathcal{D}_\theta$. □

The set $\mathcal{D}_\theta^s$ is defined by piecewise linear constraints and, although generally nonconvex, it is the union of at most 2^p convex set $\mathcal{D}_{\theta i}^s$, that is

$$\mathcal{D}_\theta^s = \bigcup_{i=1}^{2^p} \mathcal{D}_{\theta i}^s, \tag{16}$$

where $\mathcal{D}_{\theta i}^s$, defined by $2N+p$ linear constraints, is the intersection between $\mathcal{D}_\theta^s$ and the ith orthant of the space $\mathbb{R}^p$. Let the relaxed Parameter uncertainty interval PUI_j^s be

$$\mathrm{PUI}_j^s = [\underline{\theta}_j^s, \overline{\theta}_j^s], \tag{17}$$

where

$$\underline{\theta}_j^s = \min_{i=1,\ldots,2^p} \underline{\theta}_{ji}^s, \tag{18}$$

$$\overline{\theta}_j^s = \max_{i=1,\ldots,2^p} \overline{\theta}_{ji}^s, \tag{19}$$

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and

$$\underline{\theta}^s_{ji} = \min_{\theta \in \mathscr{D}^s_{\theta i}} \theta_j, \tag{20}$$

$$\overline{\theta}^s_{ji} = \max_{\theta \in \mathscr{D}^s_{\theta i}} \theta_j. \tag{21}$$

Note that, since ξ and η are not optimization variables of problems (20) and (21) and $\mathscr{D}^s_\theta$ is an outer approximation of $\mathscr{D}_\theta$, then

$$\underline{\theta}^s_j \leqslant \underline{\theta}_j \quad \text{for all } j = 1, \ldots, p, \tag{22}$$

$$\overline{\theta}^s_j \geqslant \overline{\theta}_j \quad \text{for all } j = 1, \ldots, p, \tag{23}$$

and

$$\mathrm{PUI}^s_j \supseteq \mathrm{PUI}_j \quad \text{for all } j = 1, \ldots, p. \tag{24}$$

Computational complexity—Computation of $\underline{\theta}^s_j$ and $\overline{\theta}^s_j$ for all $j = 1, \ldots, p$ requires the solution to $2p2^p$ LP optimization problems, because problems (20)–(21) must be solved for each of the p parameters θ_j and in all sets $\mathscr{D}^s_{\theta i}$ for $i = 1 \ldots 2^p$.

4. SEMI-STATIC EIV RELAXATION

In this section we present a new technique to relax (12)–(13) to convex optimization problems that can be solved using Interior Point algorithms. For the sake of clarity, a general overview of the proposed method is first presented in Section 4.1. Then, detailed technical results are provided in Section 4.2.

4.1. Overview of the method

A key feature of the static EIV relaxation method reviewed in Section 3 is that the number of optimization variables involved in the computation of the relaxed bounds does not depend on the length of the data sequence. On the contrary, this is not true for the identification problems (12)–(13) where each noise sample enters the problem as an optimization variable. However, since independence of the optimization space dimension on the number of measurements N is obtained by neglecting the correlation among successive uncertain regressors, relaxed intervals PUI^s_j may result, in general, quite conservative. The aim of the semi-static EIV relaxation presented in this work is to reduce the conservativeness of the static EIV method without increasing the number of optimization variables. In order to achieve this objective, we first provide an alternative description of the FPS $\mathscr{D}_\theta$ as the intersection of N sets $\mathscr{S}^n_k$, each one obtained as follows:

(i) given a fixed integer $n \in [1, N]$, we consider the equality constraints in (9) corresponding to a finite number n of consecutive measurements;
(ii) we substitute the first constraint in the second one obtaining a new equality that retains correlations among the two; then, the newly obtained equation is substituted into the third constraint in order to retain correlation among the first three equalities selected in (i); the procedure is repeated until all the n consecutive constraints are nested.

Then, the same approach exploited in the static EIV relaxation is applied to the derived alternative description of $\mathscr{D}_\theta$ in order to obtain a new outer-bounding set $\mathscr{D}^{ss(n)}_\theta$. Such a set is then used to compute the new relaxed interval $\mathrm{PUI}^{ss(n)}$. Like in the static EIV relaxation, the computation of $\mathrm{PUI}^{ss(n)}$ requires the solution to optimization problems where the variables are the p unknown parameters θ. However, differently from the static EIV, in the semi-static EIV the correlation between consecutive measurements is not completely lost in the construction of the approximated

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set $\mathscr{D}_\theta^{ss(n)}$. In fact, while in the static EIV all the N regressors are considered independent of each other, in the semi-static EIV the outer-bound $\mathscr{D}_\theta^{ss(n)}$ is constructed by taking into account the correlation between n consecutive measurements.

Thanks to the structure of the $\mathscr{D}_\theta^{ss(n)}$, which is the union of a finite number of semialgebraic sets, the computation of the PUI$^{ss(n)}$ is performed by exploiting LMI-relaxation techniques.

4.2. *Technical results*

Technical details of the proposed method are discussed in this section. All the proofs of the results and properties presented in the paper can be found in the Appendix.

Results 2 and 3 below provide two alternative descriptions of the FPS $\mathscr{D}_\theta$, which will play a key role in the proposed approach.

Result 2—Alternative description of the FPS

Given a fixed integer $n \in [1, N]$, let us define the set $\mathscr{S}_k^{(n)}$ as

$$\mathscr{S}_k^{(n)} = \{\theta \in \mathbb{R}^p : A(q^{-1})(y_t - \eta_t) = B(q^{-1})(u_t - \xi_t),$$
$$|\xi_t| \leqslant \Delta\xi_t,\ |\eta_t| \leqslant \Delta\eta_t;\ t = k, k+1, \ldots, \min\{k+n-1, N\}\}. \tag{25}$$

Then, $\mathscr{D}_\theta$ can be written as

$$\mathscr{D}_\theta = \bigcap_{k=1}^{N} \mathscr{S}_k^{(n)}. \tag{26}$$

□

Thanks to Result 2, the FPS $\mathscr{D}_\theta$ is written as the intersection of N sets $\mathscr{S}_k^{(n)}$, each one defined by the constraints describing $\mathscr{D}_\theta$ in (9) obtained by at most n consecutive measurements. For instance, $\mathscr{S}_1^{(n)}$ is defined by the measurement constraints from $t=1$ up to $t=n$, $\mathscr{S}_2^{(n)}$ is defined by measurement constraints from $t=2$ up to $t=n+1$, and so on, up to $\mathscr{S}_N^{(n)}$ that will be defined just by measurement constraint at $t=N$. In this work, we refer to n as the *dynamic horizon*.

We point out that the generic constraint $A(q^{-1})(y_t - \eta_t) = B(q^{-1})(u_t - \xi_t)$ in (9) and (25) can be written in the matrix form

$$\underline{y}_t = F(\underline{y}_{t-1} - \underline{\eta}_{t-1}) + G(\underline{u}_t - \underline{\xi}_t) + \underline{\eta}_t, \tag{27}$$

$$y_t = [1\ 0\ \ldots\ 0]\underline{y}_t, \tag{28}$$

where

$$\underline{y}_t \in \mathbb{R}^{na} : \underline{y}_t^{\mathrm{T}} = [y_t, y_{t-1}, \ldots, y_{t-na+1}], \tag{29}$$

$$\underline{u}_t \in \mathbb{R}^{nb+1} : \underline{u}_t^{\mathrm{T}} = [u_t, u_{t-1}, \ldots, u_{t-nb}], \tag{30}$$

$$\underline{\eta}_t \in \mathbb{R}^{na} : \underline{\eta}_t^{\mathrm{T}} = [\eta_t, \eta_{t-1}, \ldots, \eta_{t-na+1}], \tag{31}$$

$$\underline{\xi}_t \in \mathbb{R}^{nb+1} : \underline{\xi}_t^{\mathrm{T}} = [\xi_t, \xi_{t-1}, \ldots, \xi_{t-nb}], \tag{32}$$

$$F \in \mathbb{R}^{na,na} : F = \begin{bmatrix} -a_1 & -a_2 & \ldots & -a_{na-1} & -a_{na} \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & . & . & . & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}, \tag{33}$$

$$G\in\mathbb{R}^{na,nb+1}:G=\begin{bmatrix} b_0 & b_1 & \dots & b_{nb-1} & b_{nb}\\ 0 & 0 & \dots & 0 & 0\\ 0 & 0 & \dots & 0 & 0\\ \vdots & . & . & . & \vdots\\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{34}$$

Then, the following alternative description of the sets $\mathscr{S}_k^{(n)}$ in terms of matrices F and G is obtained.

Result 3—Alternative description of $\mathscr{S}_k^{(n)}$
The set $\mathscr{S}_k^{(n)}$ defined in (25) can be written as

$$\mathscr{S}_k^{(n)}=\{\theta\in\mathbb{R}^p:y_{k+r-1}=[1\ \ 0\dots 0]\left[F^r(\underline{y}_{k-1}-\underline{\eta}_{k-1})+\sum_{j=1}^{r}F^{r-j}G(\underline{u}_{k+r-j}-\underline{\xi}_{k+r-j})+\underline{\eta}_{k+r-1}\right],$$
$$|\xi_t|\leqslant\Delta\xi_t,\quad |\eta_t|\leqslant\Delta\eta_t,\quad r=1,\dots,\min\{n,N-k+1\},$$
$$t=k,k+1,\dots,\min\{k+n-1,N\}\}, \tag{35}$$

where F^0 is the identity matrix of dimension na. □

The set $\mathscr{S}_k^{(n)}$ in (35) can be rewritten in the form

$$\mathscr{S}_k^{(n)}=\{\theta\in\mathbb{R}^p:y_{k+r-1}=\sum_{i=1}^{na}a_i^{(r)}(y_{k-i}-\eta_{k-i})+\sum_{j=1}^{nb+r}b_j^{(r)}(u_{k+r-j}-\xi_{k+r-j})+\eta_{k+r-1},$$
$$|\xi_t|\leqslant\Delta\xi_t,\ \ |\eta_t|\leqslant\Delta\eta_t,$$
$$r=1,\dots,\min\{n,N-k+1\},\ \ t=k,k+1,\dots,\min\{k+n-1,N\}\}, \tag{36}$$

where coefficients $a_i^{(r)}, b_j^{(r)}, r=1,\dots,\min\{n,N-k+1\}$ are polynomial functions of the unknown parameters θ_j. In particular $a_i^{(r)}=[F^r](1,i)$ and $b_j^{(r)}=\sum_{i=1}^{r}([F^{r-i}G](1,j-i+1))$, where $[R](1,k)$ is the element in the first row and in the kth column of a generic matrix R if $k>0$, whereas $[R](1,k)=0$ when $k\leqslant 0$.

Remark 1
The maximum degree of the polynomials $a_i^{(r)}$ and $b_j^{(r)}$, with $i=1,\dots,na$ and $j=1,\dots,nb+r$, is equal to r. Therefore, since the maximum value taken by r is n, we can conclude that the degree of the polynomials $a_i^{(r)}$ and $b_j^{(r)}$ is always less than or equal to the *dynamic horizon* n.

It must be pointed out that Result 2 provides only an alternative description of the exact FPS $\mathscr{D}_\theta$. Outer-bounds $\mathscr{S}_k^{ss(n)}$ and $\mathscr{D}_\theta^{ss(n)}$ of $\mathscr{S}_k^{(n)}$ and $\mathscr{D}_\theta$, respectively, are then constructed as described in the following results. Such outer-bounds allow the evaluation of parameter bounds by solving semialgebraic optimization problems such as (12) and (13), with p variables instead of $p+2N$.

Result 4—Construction of the outer-bound $S_k^{ss(n)}$
Let us define the set $\mathscr{S}_k^{ss(n)}$ as

$$\mathscr{S}_k^{ss(n)}=\{\theta\in\mathbb{R}^p:(\varphi_k^{(r)}-\overline{\Delta\varphi_k^{(r)}})^{\mathrm{T}}q^{(r)}\leqslant y_{k+r-1}+\Delta y_{k+r-1},$$
$$(\varphi_k^{(r)}+\overline{\Delta\varphi_k^{(r)}})^{\mathrm{T}}q^{(r)}\geqslant y_{k+r-1}-\Delta y_{k+r-1},$$
$$r=1,\dots,\min\{n,N-k+1\}\}, \tag{37}$$

where

$$\varphi_k^{(r)} \in \mathbb{R}^{p+r-1} = [y_{k-1}, \ldots, y_{k-na}, u_{k+r-1}, \ldots, u_{k-nb+1}, u_{k-nb}]^{\mathrm{T}}, \tag{38}$$

$$q^{(r)} \in \mathbb{R}^{p+r-1} = [a_1^{(r)}, \ldots, a_{na}^{(r)}, b_1^{(r)}, \ldots, b_{nb+r-1}^{(r)}, b_{nb+r}^{(r)}]^{\mathrm{T}}, \tag{39}$$

$$\overline{\Delta\varphi_k^{(r)}} \in \mathbb{R}^{p+r-1} = [\Delta\eta_{k-1}\,\mathrm{sign}(a_1^{(r)}), \ldots, \Delta\eta_{k-na}\,\mathrm{sign}(a_{na}^{(r)}), \Delta\xi_{k+r-1}\,\mathrm{sign}(b_1^{(r)}), \Delta\xi_{k+r-2}\,\mathrm{sign}(b_2^{(r)}), \ldots, \Delta\xi_{k-nb}\,\mathrm{sign}(b_{nb+r}^{(r)})]^{\mathrm{T}}. \tag{40}$$

The set $\mathscr{S}_k^{ss(n)}$ is an outer approximation of $\mathscr{S}_k^{(n)}$, that is

$$\mathscr{S}_k^{ss(n)} \supseteq \mathscr{S}_k^{(n)} \quad \text{for all } k=1,\ldots,N. \tag{41}$$

□

Remark 2

The number of components of the vector $q^{(r)}$ is $p+r-1$. Let q_n be the collection of the vectors $q^{(r)}$, for $r=1,\ldots,n$, that is

$$q_n = \begin{bmatrix} q^{(1)} \\ q^{(2)} \\ \vdots \\ q^{(n)} \end{bmatrix}. \tag{42}$$

The maximum amount of diverse combinations of polynomial functions in the parameters θ_j, $j=1,\ldots,p$, in (37) is equal to the sum of the number of the components of all the vectors $q^{(r)}$, for $r=1,\ldots,n$, or equivalently, is equal to the number of components of q_n, that is $\sum_{r=1}^{n} p+r-1 = n(p+(n-1)/2)$.

An outer-bound of the FPS $\mathscr{D}_\theta$ can be constructed as described by the following result.

Result 5—Construction of the relaxed FPS $\mathscr{D}_\theta^{ss}$

Let us define the set $\mathscr{D}_\theta^{ss(n)}$ as:

$$\mathscr{D}_\theta^{ss(n)} = \bigcap_{k=1}^{N} \mathscr{S}_k^{ss(n)}. \tag{43}$$

The set $\mathscr{D}_\theta^{ss(n)}$ is an outer-bound of $\mathscr{D}_\theta$, that is

$$\mathscr{D}_\theta^{ss(n)} \supseteq \mathscr{D}_\theta. \tag{44}$$

□

The relaxed intervals $\mathrm{PUI}_j^{ss(n)}$ for a dynamic horizon n, defined as

$$\mathrm{PUI}_j^{ss(n)} = [\underline{\theta}_j^{ss(n)}, \overline{\theta}_j^{ss(n)}], \tag{45}$$

can be evaluated by solving the optimization problems

$$\underline{\theta}_j^{ss(n)} = \min_{\theta \in \mathscr{D}_\theta^{ss(n)}} \theta_j \quad \text{for } j=1,\ldots,p \tag{46}$$

$$\overline{\theta}_j^{ss(n)} = \max_{\theta \in \mathscr{D}_\theta^{ss(n)}} \theta_j \quad \text{for } j=1,\ldots,p. \tag{47}$$

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Remark 3
The optimization problems to be solved for the computation of $\mathrm{PUI}_j^{ss(n)}$ and the PUI_j^s involve the same p variables, i.e. the unknown components of vector θ. However, while in the static EIV all the N regressors are considered independent of each other, in the semi-static EIV the outer-bound $\mathscr{S}_k^{ss(n)}$ of the set $\mathscr{S}_k^{(n)}$ is constructed by considering the correlation between n consecutive measurements through nested substitution of the equations that describe $\mathscr{S}_k^{(n)}$. Besides, when the dynamic horizon n is equal to 1, the two constraints defining $\mathscr{S}_k^{ss(1)}$ are exactly the two inequalities, for $t=k$, in the description of the set $\mathscr{D}_\theta^s$. Therefore, from the definition of $\mathscr{D}_\theta^s$ and $\mathscr{D}_\theta^{ss(n)}$ in (14) and (43), respectively, it follows:

$$\mathscr{D}_\theta^s = \bigcap_{k=1}^{N} \mathscr{S}_k^{ss(1)} = \mathscr{D}_\theta^{ss(1)}. \tag{48}$$

Then, optimization problems (20) and (21) provide the same solution of (46) and (47) when $n=1$, that is:

$$\underline{\theta}_j^s = \underline{\theta}_j^{ss(1)} \quad \text{for } j=1,\ldots,p, \tag{49}$$

$$\overline{\theta}_j^s = \overline{\theta}_j^{ss(1)} \quad \text{for } j=1,\ldots,p, \tag{50}$$

or equivalently

$$\mathrm{PUI}_j^s = \mathrm{PUI}_j^{ss(1)} \quad \text{for } j=1,\ldots,p. \tag{51}$$

This means that the static EIV can be seen as a particular case of the semi-static EIV relaxation.

Evaluation of parameter bounds $\mathrm{PUI}_j^{ss(n)}$ requires the solution to problems (46)–(47) over the nonconvex feasible region $\mathscr{D}_\theta^{ss(n)}$. In the following we describe how to find a solution to these problems, exploiting the particular structure of $\mathscr{D}_\theta^{ss(n)}$ provided by Property 1. In order to analyze the topological features of the set $\mathscr{D}_\theta^{ss(n)}$, the following definitions are given. Let Γ be the set of all vectors with $n(p+(n-1)/2)$ components, each one equal to ± 1. This means that $\Gamma = \{\alpha_1, \alpha_2, \ldots, \alpha_z, \ldots, \alpha_Z\}$, where $Z = 2^{n(p+(n-1)/2)}$ and α_z is a vector with $n(p+(n-1)/2)$ components, each one equal to ± 1 and such that:

$$\alpha_z \neq \alpha_i \quad \text{if } z \neq i \quad \text{for all } z, i = 1, \ldots, Z. \tag{52}$$

Denote $\mathscr{O}(\alpha_z)$ the subset of $\mathbb{R}^p$ as

$$\mathscr{O}(\alpha_z) = \left\{\theta \in \mathbb{R}^p : \alpha_{zj} q_{nj}(\theta) \geqslant 0,\ j=1,\ldots,n\left(p+\frac{n-1}{2}\right)\right\}, \tag{53}$$

where α_z belongs to Γ, and α_{zj} and q_{nj} are the jth element of the vectors α_z and q_n, respectively. Note that

$$\mathbb{R}^p = \bigcup_{z=1}^{Z} \mathscr{O}(\alpha_z). \tag{54}$$

Topological features of $\mathscr{D}_\theta^{ss(n)}$ are highlighted by Property 1.

Property 1—Structure of the set $\mathscr{D}_\theta^{ss(n)}$
The set $\mathscr{D}_\theta^{ss(n)}$ is the union of at most Z sets $\mathscr{D}_{\theta z}^{ss(n)}$ in $\mathbb{R}^p$, that is

$$\mathscr{D}_\theta^{ss(n)} = \bigcup_{z=1}^{Z} \mathscr{D}_{\theta z}^{ss(n)}, \tag{55}$$

where

$$\mathscr{D}_{\theta z}^{ss(n)} = \mathscr{D}_{\theta}^{ss(n)} \cap \mathscr{O}(\alpha_z). \tag{56}$$

Furthermore, each $\mathscr{D}_{\theta z}^{ss(n)}$, if not empty, is a semialgebraic set over $\mathbb{R}^p$. □

Remark 4

From Remark 1 the maximum degree over $r = 1, \ldots \min\{n, N-k+1\}$ of the polynomials $q_j^{(r)}$ is less than or equal to n. This means that the constraints describing $S_k^{ss(n)}$ and $\mathscr{D}_{\theta z}^{ss(n)}$ (with $z = 1, \ldots, Z$) are polynomial inequalities with maximum degree n.

The following result shows that (46) and (47) can be decomposed into a collection of optimization problems.

Result 6—Decomposition of optimization problems (46) and (47)

Solving (46)–(47) is equivalent to finding the solution to the following problems:

$$\underline{\theta}_j^{ss(n)} = \min_{z=1,\ldots,Z} \underline{\theta}_{jz}^{ss(n)}, \tag{57}$$

$$\overline{\theta}_j^{ss(n)} = \max_{z=1,\ldots,Z} \overline{\theta}_{jz}^{ss(n)}, \tag{58}$$

where

$$\underline{\theta}_{jz}^{ss(n)} = \min_{\theta \in \mathscr{D}_{\theta z}^{ss(n)}} \theta_j, \quad z = 1, \ldots, Z, \tag{59}$$

$$\overline{\theta}_{jz}^{ss(n)} = \max_{\theta \in \mathscr{D}_{\theta z}^{ss(n)}} \theta_j, \quad z = 1, \ldots, Z. \tag{60}$$

□

The decomposition of $\mathscr{D}_{\theta}^{ss(n)}$ into a union of semialgebraic sets allows us to split problems (46) and (47) into Z constrained polynomial optimization problems (59) and (60) over the parameter space $\mathbb{R}^p$. Therefore, since the number of optimization variables involved in problems (59) and (60) is small, being equal to the dimension of θ, computation of relaxed solution to such polynomial problems by means of LMI-relaxation techniques given in [24, 25] is computationally tractable. In particular, for a given relaxation order δ, application of the method proposed in [25] to problems (59) and (60) leads to approximate solutions $\underline{\theta}_{jz}^{ss(n)}(\delta)$ and $\overline{\theta}_{jz}^{ss(n)}(\delta)$ of $\underline{\theta}_{jz}^{ss(n)}$ and $\overline{\theta}_{jz}^{ss(n)}$, respectively, computed by solving the semidefinite programming (SDP) problems

$$\underline{\theta}_{jz}^{ss(n)}(\delta) = \min_{x \in \mathscr{D}_{xz}^{ss(n,\delta)}} f_j(x), \tag{61}$$

$$\overline{\theta}_{jz}^{ss(n)}(\delta) = \max_{x \in \mathscr{D}_{xz}^{ss(n,\delta)}} f_j(x), \tag{62}$$

where the variables in the vector x are called LMI decision variables, unlike the optimization variables of problems (59)–(60) which are called polynomial variables. The function $f_j(x)$ is linear and $\mathscr{D}_{xz}^{ss(n,\delta)}$ is a convex set defined by LMI that takes into account the constraints characterizing $\mathscr{D}_{\theta z}^{ss(n)}$. It must be pointed out that the minimum allowed value $\underline{\delta}$ of the LMI-relaxation order is $\lceil \rho(\mathscr{D}_{\theta z}^{ss(n)})/2 \rceil$, where $\rho(\mathscr{D}_{\theta z}^{ss(n)})$ is the maximum degree of the polynomial constraints defining $\mathscr{D}_{\theta z}^{ss(n)}$ and $\lceil a \rceil$ is the smallest integer greater than or equal to a. From Remark 1 the maximum degree of the polynomial inequalities that describe $\mathscr{D}_{\theta z}^{ss(n)}$ is always less than or equal to n; therefore, for relaxation order $\delta \geqslant \underline{\delta} = \lceil n/2 \rceil$, problems (61) and (62) are well defined. See Lasserre [25] for technical details on the relaxation of semialgebraic optimization problems to convex SDP problems.

A MATLAB implementation of this relaxation technique, developed by Henrion and Lasserre, can be found in the open source freeware software Gloptipoly [28], which exploits the LMI solver SeDuMi [29] to solve semidefinite optimization problems in polynomial time.

Let us define the relaxed uncertainty interval $\mathrm{PUI}_j^{ss(n)}(\delta)$ obtained for a relaxation order δ as

$$\mathrm{PUI}_j^{ss(n)}(\delta) = [\underline{\theta}_j^{ss(n)}(\delta); \overline{\theta}_j^{ss(n)}(\delta)], \tag{63}$$

where

$$\underline{\theta}_j^{ss(n)}(\delta) = \min_{z=1,\dots,Z} \underline{\theta}_{jz}^{ss(n)}(\delta) \quad \text{for } j=1,\dots,p, \tag{64}$$

$$\overline{\theta}_j^{ss(n)}(\delta) = \max_{z=1,\dots,Z} \overline{\theta}_{jz}^{ss(n)}(\delta) \quad \text{for } j=1,\dots,p. \tag{65}$$

Then, the following property holds.

Property 2—Monotone convergence to parameter uncertainty intervals $\mathrm{PUI}_j^{ss(n)}$
For any $\delta \geqslant \underline{\delta}$, the relaxed interval $\mathrm{PUI}_j^{ss}(\delta)$ becomes tighter as the LMI-relaxation order δ increases, that is:

$$\mathrm{PUI}_j^{ss(n)}(\delta+1) \subseteq \mathrm{PUI}_j^{ss(n)}(\delta). \tag{66}$$

Besides, relaxed intervals $\mathrm{PUI}_j^{ss(n)}(\delta)$ converge to $\mathrm{PUI}_j^{ss(n)}$ as the LMI-relaxation order approaches infinity, that is:

$$\lim_{\delta\to\infty} \underline{\theta}_j^{ss(n)}(\delta) = \underline{\theta}_j^{ss(n)}, \tag{67}$$

$$\lim_{\delta\to\infty} \overline{\theta}_j^{ss(n)}(\delta) = \overline{\theta}_j^{ss(n)}. \tag{68}$$

□

Remark 5
As stated in Remark 3, $\mathrm{PUI}_j^{s} = \mathrm{PUI}_j^{ss(1)}$. Therefore, $\mathrm{PUI}_j^{ss(1)}$ can be computed by solving the LP problems (20) and (21) instead of exploiting LMI-relaxation techniques. Thus, $\mathrm{PUI}_j^{ss(1)}$ does not depend on the relaxation order δ, that is:

$$\mathrm{PUI}_j^{s} = \mathrm{PUI}_j^{ss(1)} = \mathrm{PUI}_j^{ss(1)}(\delta) \quad \text{for all } j=1,\dots,p, \quad \delta \geqslant \underline{\delta}. \tag{69}$$

5. ESTIMATE ACCURACY AND COMPUTATIONAL COMPLEXITY ANALYSIS

In this section, we analyze the estimate accuracy and the computational complexity in the evaluation of parameter bounds $\mathrm{PUI}_j^{ss(n)}$.

5.1. Estimate accuracy

There is no guarantee that the relaxed parameter bounds $\mathrm{PUI}_j^{ss(n)}$ converge to the tight bounds PUI_j. In fact, as in the static EIV case, the relaxation of the set $\mathscr{S}_k^{(n)}$ to $\mathscr{S}_k^{ss(n)}$, described in Result 5, is carried out by assuming that the noise samples affecting successive equations in the definition of $\mathscr{S}_k^{(n)}$ in (36) vary independently, with a consequent loss of the information on the correlation between consecutive measurements. However, some information on the correlation between n consecutive measurements is retained by the semi-static EIV relaxation. In fact, while in the static EIV all the N constraints defining the feasible set $\mathscr{D}_\theta$ are considered as independent, in the semi-static EIV the set $\mathscr{S}_k^{(n)}$ is relaxed to $\mathscr{S}_k^{ss(n)}$ after the correlation between n consecutive

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measurements is retained through nested substitution of the equality constraints that describe $\mathscr{S}_k^{(n)}$. The following property shows that parameter intervals $\mathrm{PUI}_j^{ss(n)}$ become tighter as the dynamic-horizon n increases.

Property 3—Dependence of parameter bounds on the dynamic horizon

For any relaxation order $\delta \geqslant \underline{\delta}$ and for any dynamic horizon $n \in [1, N-1]$, $\mathrm{PUI}_j^{ss(n+1)}(\delta)$ are always included in the $\mathrm{PUI}_j^{ss(n)}(\delta)$, that is:

$$\mathrm{PUI}_j^{ss(n)}(\delta) \supseteq \mathrm{PUI}_j^{ss(n+1)}(\delta) \quad \text{for all } j = 1, \ldots, p. \tag{70}$$

□

Note that the static EIV (or, equivalently, the semi-static relaxation for $n=1$) provides a worse estimate, in term of accuracy, of the parameter bounds with respect to a semi-static relaxation carried out with dynamic horizon greater than 1. In fact, as previously discussed, no information on the correlation between consecutive measurements is retained in the static EIV relaxation.

5.2. Computational complexity of parameter bounds evaluation

Property 4 argues on the number of optimization problems to be solved in order to evaluate the relaxed intervals $\mathrm{PUI}_j^{ss(n)}$.

Property 4—Number of optimization problems

The computation of $\mathrm{PUI}_j^{ss(n)}$ for all $j = 1, \ldots, p$ requires the solution to at most $2p2^{n(p+(n-1)/2)}$ constrained polynomial optimization problems, which means that the number of optimization problems to be solved in order to compute the PUIs grows.

(P4.1) exponentially with the dynamic horizon n,
(P4.2) exponentially with the number p of unknown parameters θ_j. □

Indeed, a deterministic correlation between the signs of some components of the vector q_n could be used to reduce the number of feasible regions $\mathscr{D}_{\theta z}^{ss(n)}$ where the optimization problems (59) and (60) have to be solved. For instance, two or more components of q_n could take the same polynomial function. Thus, the regions $\mathscr{D}_{\theta z}^{ss(n)}$, where the signs of the components of q_n are different, are empty set and problems (59) and (60) cannot be solved in such regions. Besides, let us suppose that a_1, b_1 and $a_1 b_1$ are components of the vector q_n. Thus, the sign of the product $a_1 b_1$ is uniquely implied by the sign of a_1 and b_1. The complexity of optimization problems (59)–(60) is discussed in the following property.

Property 5—Computational complexity of optimization problems (59)–(60)

The constrained polynomial optimization problems (59)–(60) enjoy the following features:

(P5.1) the number of optimization variables is equal to p,
(P5.2) the number of polynomial constraints defining the relaxed feasible set $\mathscr{D}_{\theta z}^{ss(n)}$ is at most $m = n(2N + p + (n-1)/2)$. □

As discussed in the paper, the approximation of global optima of polynomial optimization problems (59)–(60) requires the use of the LMI relaxation proposed in [25], whose complexity is given by the following property.

Property 6—Complexity of relaxed problems (61)–(62)

(P6.1) The number of LMI decision variables of SDP problems (61) and (62) is

- $O(\delta^p)$ for fixed p,
- $O(p^{2\delta})$ for fixed δ,
- independent on n.

(P6.2) The size of LMI constraint of SDP problems (61)–(62) is

- $O(N(\delta)^p)$ for fixed p and n,

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- $O(p^{\delta}+Np^{\delta-1})$ for fixed δ and n,
- $O(n^2+nN)$ for fixed p and δ.

6. SIMULATED EXAMPLE

In this section, we illustrate the proposed parameter bounding procedure through a simulated example. The numerical computation is carried out on a 2.40-GHz Intel Pentium IV with 3 GB of RAM. PUIs are computed by exploiting the static EIV (or equivalently the semi-static EIV for a dynamic horizon $n=1$) and the semi-static EIV with two different values of the dynamic horizon. The considered system is characterized by (1), (2) and (3) with: $A(q^{-1})=(1-1.12q^{-1}+0.88q^{-2})$ and $B(q^{-1})=(1.54q^{-1}-1.52q^{-2})$. Thus, the true parameter vector is $\theta^{\mathrm{T}}=[a_1\ a_2\ b_1\ b_2]=[-1.12\ 0.88\ 1.54\ -1.52]$. The system is excited by a random input sequence x_t uniformly distributed between $[-1,\ +1]$. Both input x_t and output sequence w_t are corrupted by random additive noise ξ_t and η_t, uniformly distributed between $[-\Delta\xi_t, +\Delta\xi_t]$ and $[-\Delta\eta_t,\ +\Delta\eta_t]$, respectively. The chosen error bounds $\Delta\xi_t$ and $\Delta\eta_t$ are such that the Signal to Noise Ratios on the input SNR_x and on the output SNR_w, defined as

$$\mathrm{SNR}_x=10\log\left\{\sum_{t=1}^{N} x_t^2 \bigg/ \sum_{t=1}^{N} \xi_t^2\right\}, \tag{71}$$

$$\mathrm{SNR}_w=10\log\left\{\sum_{t=1}^{N} w_t^2 \bigg/ \sum_{t=1}^{N} \eta_t^2\right\}, \tag{72}$$

are equal to 32 and 28 db, respectively. The number of measurements N exploited to compute the parameter bounds is equal to 300. First, the static EIV approach is used to compute $\underline{\theta}_j^s$ and $\overline{\theta}_j^s$, which define the PUI_j^s. The obtained relaxed bounds $\overline{\theta}_j^s$ and $\underline{\theta}_j^s$, the central estimate θ_j^{cs} and the parameter uncertainty bounds $\Delta\theta_j^s$, defined as

$$\theta_j^{cs}=\frac{\overline{\theta}_j^s+\underline{\theta}_j^s}{2}, \tag{73}$$

$$\Delta\theta_j^s=\frac{\overline{\theta}_j^s-\underline{\theta}_j^s}{2}, \tag{74}$$

are reported in Table I. The width of the interval PUI_j^s is given by $2\Delta\theta_j^s$. The elapsed time to compute a single parameter bound ($\underline{\theta}_j^s$ or $\overline{\theta}_j^s$) using the linprog function of the Matlab optimization toolbox is between 0.4 and 0.5 s.

Then, semi-static EIV is exploited to evaluate the PUI_j^{ss}. The chosen values of the dynamic-horizon are 2 and 3, whereas the LMI-relaxation order δ is chosen equal to 2. The obtained values

Table I. Static EIV relaxation—Parameter central estimates (θ_j^{cs}), parameter bounds ($\underline{\theta}_j^s$, $\overline{\theta}_j^s$) and parameter uncertainty bounds $\Delta\theta_j^s$.

True value	$\underline{\theta}_j^s$	θ_j^{cs}	$\overline{\theta}_j^s$	$\Delta\theta_j^s$
−1.1200	−1.3571	−1.1409	−0.9248	0.2162
0.8800	0.7308	0.9047	1.0785	0.1738
1.5400	0.7289	1.5416	2.3543	0.8127
−1.4200	−2.2337	−1.3884	−0.5431	0.8453

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Table II. Semi-static EIV relaxation, $n=2$—Parameter central estimates ($\theta_j^{css(2)}(2)$), parameter bounds ($\underline{\theta}_j^{ss(2)}(2)$, $\overline{\theta}_j^{ss(2)}(2)$) and parameter uncertainty bounds $\Delta\theta_j^{ss(2)}(2)$.

True value	$\underline{\theta}_j^{ss(2)}(2)$	$\theta_j^{css(2)}(2)$	$\overline{\theta}_j^{ss(2)}(2)$	$\Delta\theta_j^{ss(2)}(2)$
−1.1200	−1.2098	−1.1205	−1.0312	0.0893
0.8800	0.7910	0.8754	0.9598	0.0844
1.5400	1.1304	1.5746	2.0187	0.4442
−1.4200	−1.9137	−1.4318	−0.9464	0.4854

Table III. Semi-static EIV relaxation, $n=3$—Parameter central estimates ($\theta_j^{css(3)}(2)$), parameter bounds ($\underline{\theta}_j^{ss(3)}(2)$, $\overline{\theta}_j^{ss(3)}(2)$) and parameter uncertainty bounds $\Delta\theta_j^{ss(3)}(2)$.

True value	$\underline{\theta}_j^{ss(3)}(2)$	$\theta_j^{css(3)}(2)$	$\overline{\theta}_j^{ss(3)}(2)$	$\Delta\theta_j^{ss(3)}(2)$
−1.1200	−1.1886	−1.1150	−1.0413	0.0737
0.8800	0.8136	0.8801	0.9466	0.0665
1.5400	1.1575	1.5471	1.9367	0.3896
−1.4200	−1.8516	−1.4267	−1.0018	0.4249

of $\underline{\theta}^{ss(n)}(2)$, $\overline{\theta}^{ss(n)}(2)$, the center $\theta_j^{ssd(n)}(2)$ and the parameter uncertainty bounds $\Delta\theta_j^{ss(n)}(2)$, which are defined as

$$\theta_j^{css(n)}(2)=\frac{\overline{\theta}_j^{ss(n)}(2)+\underline{\theta}_j^{ss(n)}(2)}{2}, \tag{75}$$

$$\Delta\theta_j^{ss(n)}(2)=\frac{\overline{\theta}_j^{ss(n)}(2)-\underline{\theta}_j^{ss(n)}(2)}{2}, \tag{76}$$

with $n=2$ and $n=3$ are reported in Tables II and III, respectively. The CPU elapsed time to evaluate through the SeDuMi solver a single parameter bound $\underline{\theta}^{ss(n)}(2)$ or $\overline{\theta}^{ss(3)}(2)$ is between 7 and 8 s when $n=2$ and between 11 and 12 s when $n=3$. A comparison of results reported in Tables I, II and III shows that a significant improvement on the accuracy of parameter bounds is obtained if the semi-static EIV (with a dynamic horizon greater than 1) is exploited instead of the static EIV. In fact, even if low values of the dynamic horizon n are used, the uncertainty bounds $\Delta\theta_j^{ss(2)}(2)$ and $\Delta\theta_j^{ss(3)}(2)$ are less than half the uncertainty bounds $\Delta\theta_j^{s}$ for each parameter θ_j. On the other hand, as expected, the elapsed CPU time to compute the PUI_j^s is significantly less than the time needed to compute both $\Delta\theta_j^{ss(2)}(2)$ and $\Delta\theta_j^{ss(3)}(2)$ exploiting the semi-static EIV. The main reason is that, in order to evaluate the PUI_j^s, LP problems are solved instead of the SDP problems needed to evaluate $\Delta\theta_j^{ss(2)}(2)$ and $\Delta\theta_j^{ss(3)}(2)$.

7. CONCLUDING REMARKS

A new procedure to evaluate parameter bounds of linear dynamic systems for set-membership EIV problems is presented. Parameter bounds evaluation is formulated as a collection of constrained polynomial optimization problems, with a number of variables that increases with the number of measurements. Therefore, because of their high computational complexity, LMI-relaxation techniques proposed in the literature to find approximate solutions of such polynomial problems can

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be exploited only if the number of measurements is small. In order to reduce the computational complexity of the identification problems, an outer-bound approximation of the feasible parameter set is sought. Since the outer-bound is the union of a finite number of semialgebraic sets over the parameter space, parameter bounds can be evaluated by solving suitable semialgebraic optimization problems involving a small number of variables, i.e. the unknown system parameters. LMI-relaxation techniques are used to approximate global optima. Finally, the capability of the proposed relaxation procedure to provide a less conservative estimate of parameter bounds with respect to the previously published static EIV is shown both theoretically and by means of a numerical example, which shows that the parameter uncertainty intervals obtained for a dynamic horizon equal to 2 and for an LMI-relaxation order equal to 2 are about half the ones obtained by exploiting the static EIV relaxation.

APPENDIX A

Proof of Result 2

From the definition of $\mathscr{S}_k^{(n)}$ in (25), the set $\mathscr{S}_k^{(1)}$ is

$$\begin{aligned}\mathscr{S}_k^{(1)} = \{\theta \in \mathbb{R}^p : A(q^{-1})(y_t - \eta_t) = B(q^{-1})(u_t - \xi_t),\\ |\xi_t| \leqslant \Delta\xi_t,\ |\eta_t| \leqslant \Delta\eta_t,\ t = k\}.\end{aligned} \tag{A1}$$

Indeed,

$$\mathscr{S}_k^{(1)} \supseteq \mathscr{S}_k^{(n)} \quad \text{for all } n = 1, \ldots, N,\ \ k = 1 \ldots N, \tag{A2}$$

and

$$\mathscr{D}_\theta = \bigcap_{k=1}^{N} \mathscr{S}_k^{(1)}. \tag{A3}$$

Then, from (A2) and (A3), the following condition holds:

$$\mathscr{D}_\theta = \bigcap_{k=1}^{N} \mathscr{S}_k^{(1)} \supseteq \bigcap_{k=1}^{N} \mathscr{S}_k^{(n)}. \tag{A4}$$

Besides, since all the constraints defining $\mathscr{S}_k^{(n)}$ are a subset of those that describe $\mathscr{D}_\theta$, it follows that

$$\mathscr{D}_\theta \subseteq \mathscr{S}_k^{(n)} \quad \text{for all } k = 1, \ldots, N, \tag{A5}$$

then

$$\mathscr{D}_\theta \subseteq \bigcap_{k=1}^{N} \mathscr{S}_k^{(n)}. \tag{A6}$$

Thus, from Equations (A4) and (A6) we can conclude that

$$\mathscr{D}_\theta = \bigcap_{k=1}^{N} \mathscr{S}_k^{(n)}. \tag{A7}$$

□

Proof of Result 3

The description of $\mathscr{S}_k^{(n)}$ given by Equation (35) is obtained by the nested substitution of the equations (27)–(28) in (25), for $t = k, k+1, \ldots, \min\{k+n-1, N\}$. □

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Proof of Result 4
First, note that the set $\mathscr{S}_k^{(n)}$ in (36) can be written in the compact form

$$\mathscr{S}_k^{(n)} = \{\theta \in \mathbb{R}^p : y_{k+r-1} = (\varphi_k^{(r)} - \delta\varphi_k^{(r)})^{\mathrm{T}} q^{(r)} + \eta_{k+r-1},$$
$$r = 1, \ldots, \min\{n, N-k+1\} \ |\delta\varphi_k^{(r)}| \leqslant \Delta\varphi_k^{(r)}\}, \tag{A8}$$

where the inequality $|\delta\varphi_k^{(r)}| \leqslant \Delta\varphi_k^{(r)}$ must be interpreted componentwise and

$$\delta\varphi_k^{(r)} \in \mathbb{R}^{p+r-1} = [\eta_{k-1}, \ldots, \eta_{k-na}, \xi_{k+r-1}, \xi_{k+r-2}, \ldots, \xi_{k-nb}]^{\mathrm{T}}, \tag{A9}$$

$$\Delta\varphi_k^{(r)} \in \mathbb{R}^{p+r-1} = [\Delta\eta_{k-1}, \ldots, \Delta\eta_{k-na}, \Delta\xi_{k+r-1}, \Delta\xi_{k+r-2}, \ldots, \Delta\xi_{k-nb}]^{\mathrm{T}}. \tag{A10}$$

Let us define the sets $\mathscr{S}_{k,r}$ and $\mathscr{S}_{k,r}^{ss}$, for $r = 1, \ldots, \min\{n, N-k+1\}$ as

$$\mathscr{S}_{k,r} = \{\theta \in \mathbb{R}^p : y_{k+r-1} = (\varphi_k^{(r)} - \delta\varphi_k^{(r)})^{\mathrm{T}} q^{(r)} + \eta_{k+r-1}, |\delta\varphi_k^{(r)}| \leqslant \Delta\varphi_k^{(r)}\}, \tag{A11}$$

$$\mathscr{S}_{k,r}^{ss} = \{\theta \in \mathbb{R}^p : (\varphi_k^{(r)} - \overline{\Delta\varphi_k^{(r)}})^{\mathrm{T}} q^{(r)} \leqslant y_{k+r-1} + \Delta y_{k+r-1}$$
$$(\varphi_k^{(r)} + \overline{\Delta\varphi_k^{(r)}})^{\mathrm{T}} q^{(r)} \geqslant y_{k+r-1} - \Delta y_{k+r-1}\}. \tag{A12}$$

Indeed

$$\mathscr{S}_k^{(n)} = \bigcap_{r=1}^{\min\{n, N-k+1\}} \mathscr{S}_{k,r}, \tag{A13}$$

$$\mathscr{S}_k^{ss(n)} = \bigcap_{r=1}^{\min\{n, N-k+1\}} \mathscr{S}_{k,r}^{ss}. \tag{A14}$$

The equality constraint describing $\mathscr{S}_{k,r}$ in (A11) is rewritten as

$$y_{k+r-1} - (\varphi_k^{(r)})^{\mathrm{T}} q^{(r)} = \eta_{k+r-1} - (\delta\varphi_k^{(r)})^{\mathrm{T}} q^{(r)}. \tag{A15}$$

Let $q_j^{(r)}$, $\delta\varphi_{kj}^{(r)}$ and $\Delta\varphi_{kj}^{(r)}$ be the jth component of the vectors $q^{(r)}$, $\delta\varphi_k^{(r)}$ and $\Delta\varphi_k^{(r)}$, respectively. By taking the absolute value in (A15), using the triangle inequality and taking into account the componentwise inequality constraints $|\delta\varphi_k^{(r)}| \leqslant \Delta\varphi_k^{(r)}$ that describe $\mathscr{S}_{k,r}$, for each $k = 1, \ldots, N$, we obtain

$$\begin{aligned}
|y_{k+r-1} - (\varphi_k^{(r)})^{\mathrm{T}} q^{(r)}| &= |\eta_{k+r-1} - (\delta\varphi_k^{(r)})^{\mathrm{T}} q^{(r)}| \\
&\leqslant |\eta_{k+r-1}| + |(\delta\varphi_k^{(r)})^{\mathrm{T}} q^{(r)}| \\
&= |\eta_{k+r-1}| + \left| \sum_{j=1}^{p+r-1} \delta\varphi_{kj}^{(r)} q_j^{(r)} \right| \\
&\leqslant |\eta_{k+r-1}| + \sum_{j=1}^{p+r-1} |\delta\varphi_{kj}^{(r)}| |q_j^{(r)}| \\
&\leqslant \Delta\eta_{k+r-1} + \sum_{j=1}^{p+r-1} \Delta\varphi_{kj}^{(r)} \operatorname{sign}(q_j^{(r)}) q_j^{(r)} \\
&= \Delta\eta_{k+r-1} + (\overline{\Delta\varphi_k^{(r)}})^{\mathrm{T}} q^{(r)}.
\end{aligned} \tag{A16}$$

Condition (A16) proves that if $q^{(r)}$ satisfies the constraints defining $\mathscr{S}_{k,r}$ in (A11), then the following inequality holds for each $k=1,\ldots,N$:

$$|y_{k+r-1}-(\varphi_k^{(r)})^{\mathrm{T}}q^{(r)}|\leqslant\Delta\eta_{k+r-1}+(\overline{\Delta\varphi_k^{(r)}})^{\mathrm{T}}q^{(r)}, \tag{A17}$$

or equivalently:

$$(\varphi_k^{(r)}-\overline{\Delta\varphi_k^{(r)}})^{\mathrm{T}}q^{(r)}\leqslant y_{k+r-1}+\Delta y_{k+r-1}, \tag{A18}$$

$$(\varphi_k^{(r)}+\overline{\Delta\varphi_k^{(r)}})^{\mathrm{T}}q^{(r)}\geqslant y_{k+r-1}-\Delta y_{k+r-1}. \tag{A19}$$

This means that if $q^{(r)}$ belongs to $\mathscr{S}_{k,r}$, then $q^{(r)}$ belongs to $\mathscr{S}_{k,r}^{ss}$, for each $k=1,\ldots,N$, that is

$$\mathscr{S}_{k,r}\subseteq\mathscr{S}_{k,r}^{ss}\quad\text{for all } r=1,\ldots,\min\{n,N-k+1\}. \tag{A20}$$

Therefore

$$\bigcap_{r=1}^{\min\{n,N-k+1\}}\mathscr{S}_{k,r}\subseteq\bigcap_{r=1}^{\min\{n,N-k+1\}}\mathscr{S}_{k,r}^{ss}\quad\text{for all } k=1,\ldots,N. \tag{A21}$$

Condition (41) follows straightforwardly from Equations (A13), (A14) and (A21).

□

Proof of Result 5
From Result 4, $\mathscr{S}_k^{ss(n)}\supseteq\mathscr{S}_k^{(n)}$, $k=1,\ldots,N$, then

$$\bigcap_{k=1}^{N}\mathscr{S}_k^{ss(n)}\supseteq\bigcap_{k=1}^{N}\mathscr{S}_k^{(n)}. \tag{A22}$$

By substituting (43) on the left side of (A22), it follows:

$$\mathscr{D}_\theta^{ss(n)}\supseteq\bigcap_{k=1}^{N}\mathscr{S}_k^{(n)}=\mathscr{D}_\theta \tag{A23}$$

that proves (44). □

Proof of Property 1
From Equation (54) and the condition $\mathscr{D}_\theta^{ss(n)}\subseteq\mathbb{R}^p$, Equation (55) follows:

$$\bigcup_{z=1}^{Z}\mathscr{D}_{\theta z}^{ss(n)}=\bigcup_{z=1}^{Z}\mathscr{D}_\theta^{ss(n)}\cap\mathscr{O}(\alpha_z)=\mathscr{D}_\theta^{ss(n)}\cap\bigcup_{z=1}^{Z}\mathscr{O}(\alpha_z)=\mathscr{D}_\theta^{ss(n)}\cap\mathbb{R}^p=\mathscr{D}_\theta^{ss(n)}. \tag{A24}$$

We are left to prove that $\mathscr{D}_{\theta z}^{ss(n)}$ is a semialgebraic set. First, we rewrite the set $\mathscr{D}_{\theta z}^{ss(n)}$ in (56) as

$$\begin{aligned}\mathscr{D}_{\theta z}^{ss(n)}=\{\theta\in\mathbb{R}^p:\,&(\varphi_k^{(r)}-\overline{\Delta\varphi_k^{(r)}})^{\mathrm{T}}q^{(r)}\leqslant y_{k+r-1}+\Delta y_{k+r-1},\\ &(\varphi_k^{(r)}+\overline{\Delta\varphi_k^{(r)}})^{\mathrm{T}}q^{(r)}\geqslant y_{k+r-1}-\Delta y_{k+r-1},\\ &\alpha_{zj}q_{nj}\geqslant 0,\\ &j=1,\ldots,n\left(p+\frac{n-1}{2}\right),\\ &k=1,\ldots,N,\\ &r=1,\ldots,\min\{n,N-k+1\}\}.\end{aligned} \tag{A25}$$

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The inequality $\alpha_{zj} q_{nj} \geqslant 0$ in (A25) means that the sign of each component q_{nj} of the vector q_n is forced and is equal to α_{zj}. Furthermore, since the components of the vector q_n are those of the vectors $q^{(r)}$, for all $r = 1, \ldots, n$, it follows that the sign of all the components of the vector $q^{(r)}$ is forced, for all $r = 1, \ldots, n$. Therefore, the vector $\overline{\Delta\varphi_k^{(r)}}$ is known for all $r = 1, \ldots, n$ and $k = 1, \ldots, N$. Then, constraints defining $\mathscr{D}_{\theta z}^{ss(n)}$ in (A25) are linear in the elements of the vector $q^{(r)}$, for all $r = 1, \ldots, \min\{n, N-k+1\}$. Since the components of the vector $q^{(r)}$ are polynomial functions of the parameter θ_j, constraints in (A25) are polynomial inequalities in the parameters θ_j, $j = 1, \ldots, p$. Therefore, $\mathscr{D}_{\theta z}^{ss(n)}$ is a semialgebraic set in $\mathbb{R}^p$.

□

Proof of Result 6
Statement of Result 6 follows directly from the fact that $\mathscr{D}_{\theta}^{ss(n)}$ can be expressed as the union of semialgebraic sets $\mathscr{D}_{\theta z}^{ss(n)}$ with $z = 1, \ldots, Z$. In fact, solving (46) and (47) over the feasible region $\mathscr{D}_{\theta}^{ss(n)}$ is equivalent to compute $\underline{\theta}_{jz}^{ss(n)}$ ($\overline{\theta}_{jz}^{ss(n)}$) over each region $\mathscr{D}_{\theta z}^{ss(n)}$ for all $z = 1, \ldots, Z$ (as carried out in problems (59) and (60)) and then to compute the minimum (maximum) over all $\underline{\theta}_{jz}^{ss(n)}$ ($\overline{\theta}_{jz}^{ss(n)}$), as carried out in (57) and (58).

□

Proof of Property 2
From the property of monotone convergence of LMI-relaxation techniques for constrained polynomial optimization problem [23, 25], the following conditions hold:

$$\underline{\theta}_{jz}^{ss(n)}(\delta) \leqslant \underline{\theta}_{jz}^{ss(n)}(\delta+1) \leqslant \underline{\theta}_{jz}^{ss(n)} \quad \text{for } j = 1, \ldots, p, \quad z = 1, \ldots, Z, \quad \delta \geqslant \underline{\delta}, \tag{A26}$$

$$\overline{\theta}_{jz}^{ss(n)}(\delta) \geqslant \overline{\theta}_{jz}^{ss(n)}(\delta+1) \geqslant \overline{\theta}_{jz}^{ss(n)} \quad \text{for } j = 1, \ldots, p, \quad z = 1, \ldots, Z, \quad \delta \geqslant \underline{\delta}. \tag{A27}$$

Besides, from the definition of $\underline{\theta}_{j}^{ss(n)}(\delta)$ in (64) and $\overline{\theta}_{j}^{ss(n)}(\delta)$ in (65) and from Equations (A26)–(A27) the conditions follow:

$$\underline{\theta}_{j}^{ss(n)}(\delta) \leqslant \underline{\theta}_{j}^{ss(n)}(\delta+1) \leqslant \underline{\theta}_{j}^{ss(n)} \quad \text{for all } j = 1, \ldots, p, \quad \delta \geqslant \underline{\delta}, \tag{A28}$$

$$\overline{\theta}_{j}^{ss(n)}(\delta) \geqslant \overline{\theta}_{j}^{ss(n)}(\delta+1) \geqslant \overline{\theta}_{j}^{ss(n)} \quad \text{for all } j = 1, \ldots, p, \quad \delta \geqslant \underline{\delta}. \tag{A29}$$

Thus, from (A28), (A29) and the definition of $\mathrm{PUI}_j^{ss(n)}(\delta)$, condition (66) holds. Besides

$$\lim_{\delta\to\infty} \underline{\theta}_{jz}^{ss(n)}(\delta) = \underline{\theta}_{jz}^{ss(n)} \quad \text{for all } j = 1, \ldots, p, \quad z = 1, \ldots, Z, \tag{A30}$$

$$\lim_{\delta\to\infty} \overline{\theta}_{jz}^{ss(n)}(\delta) = \overline{\theta}_{jz}^{ss(n)} \quad \text{for all } j = 1, \ldots, p, \quad z = 1, \ldots, Z. \tag{A31}$$

Therefore, from Equations (A30), (A31) and the definition of $\underline{\theta}_{j}^{ss(n)}(\delta)$–$\overline{\theta}_{j}^{ss(n)}(\delta)$, we obtain

$$\begin{aligned}\lim_{\delta\to\infty} \underline{\theta}_{j}^{ss(n)}(\delta) &= \lim_{\delta\to\infty} \min_{z=1,\ldots,Z} \underline{\theta}_{jz}^{ss(n)}(\delta)\\ &= \min_{z=1,\ldots,Z} \underline{\theta}_{jz}^{ss(n)} = \underline{\theta}_{j}^{ss(n)} \quad \text{for all } j = 1, \ldots, p,\end{aligned} \tag{A32}$$

$$\begin{aligned}\lim_{\delta\to\infty} \overline{\theta}_{j}^{ss(n)}(\delta) &= \lim_{\delta\to\infty} \max_{z=1,\ldots,Z} \overline{\theta}_{jz}^{ss(n)}(\delta)\\ &= \max_{z=1,\ldots,Z} \overline{\theta}_{jz}^{ss(n)} = \overline{\theta}_{j}^{ss(n)} \quad \text{for all } j = 1, \ldots, p.\end{aligned} \tag{A33}$$

Therefore, from Equations (A32) and (A33), conditions (67) and (68) follow. □

Proof of Property 3
Since all the constraints defining $\mathscr{S}_k^{ss(n)}$ in (37) are also constraints describing the set $\mathscr{S}_k^{ss(n+1)}$, we can state that

$$\mathscr{S}_k^{ss(n)} \supseteq \mathscr{S}_k^{ss(n+1)} \quad \text{for all } n=1,\ldots,N-1, \;\; k=1,\ldots,N. \tag{A34}$$

Furthermore, from condition (A34) and the definition of $\mathscr{D}_\theta^{ss(n)}$ in Equation (43), the following condition holds:

$$\mathscr{D}_\theta^{ss(n)} \supseteq \mathscr{D}_\theta^{ss(n+1)} \quad \text{for all } \; n=1,\ldots,N-1. \tag{A35}$$

Besides, since the constraints describing $\mathscr{D}_{\theta z}^{ss(n)}$ are also constraints that define $\mathscr{D}_{\theta z}^{ss(n+1)}$, then, from the properties of LMI relaxation in [25], all the LMIs constraints defining $\mathscr{D}_{xz}^{ss(n,\delta)}$ also describe $\mathscr{D}_{xz}^{ss(n+1,\delta)}$. This means that

$$\mathscr{D}_{xz}^{ss(n,\delta)} \supseteq \mathscr{D}_{xz}^{ss(n+1,\delta)} \quad \text{for all } n=1,\ldots,N-1, \;\; z=1,\ldots,Z, \;\; \delta \geqslant \underline{\delta}. \tag{A36}$$

Thus, the following conditions hold:

$$\underline{\theta}_j^{ss(n)}(\delta) \leqslant \underline{\theta}_j^{ss(n+1)}(\delta) \quad \text{for all } \; j=1,\ldots,p, \;\; n=1,\ldots,N-1, \;\; \delta \geqslant \underline{\delta}, \tag{A37}$$

$$\overline{\theta}_j^{ss(n)}(\delta) \geqslant \overline{\theta}_j^{ss(n+1)}(\delta) \quad \text{for all } \; j=1,\ldots,p, \;\; n=1,\ldots,N-1, \;\; \delta \geqslant \underline{\delta}. \tag{A38}$$

From (A37)–(A38) and the definition of $\mathrm{PUI}_j^{ss(n)}(\delta)$, condition (70) follows. □

Proof of Property 4
From Result 6, the evaluation of parameter bounds $\mathrm{PUI}_j^{ss(n)}$ requires the solution to the minimization problem (59) and the maximization problem (60) in at most $Z=2^{n(p+(n-1)/2)}$ feasible regions $\mathscr{D}_{\theta z}^{ss(n)}$. Since such optimization problems have to be solved for all $j=1,\ldots,p$, the total number of constrained polynomial optimization problems to be solved is $2p2^{n(p+(n-1)/2)}$. Statements (P4.1) and (P4.2) follow. □

Proof of Property 5
Statement (P5.1) follows directly from the definition of problems (59)–(60). Besides, the number of constraints of the kind $\alpha_{zj}q_{nj}$ in the definition of $\mathscr{D}_{\theta z}^{ss(n)}$ in (A25) is $Q=n(p+(n-1)/2)$, whereas the number of the remaining constraints is at most $L=2nN$ since $r=1,\ldots,\min\{n, N-k+1\}$ and $k=1,\ldots,N$. Property (P5.2) follows by computing $m=Q+L$. □

Proof of Property 6
As discussed in Property 5, problems (59)–(60) have p optimization variables and m polynomial inequality constraints. Then, the use of the theory of moments to relax problems (59) and (60) leads to a number of LMI decision variables equal to

$$\binom{p+2\delta}{p}$$

and to an LMI constraint whose size is

$$\binom{p+\delta}{p} + m\binom{p+\delta-1}{p}.$$

Statements (P6.1) and (P6.2) follow. □

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