A unified framework for deterministic and probabilistic $\mathscr{D}$-stability analysis of uncertain polynomial matrices

Dario Piga*, Alessio Benavoli*

***Abstract*—In control theory, we are often interested in robust $\mathscr{D}$-stability analysis, which aims at verifying if all the eigenvalues of an uncertain matrix lie in a given region $\mathscr{D}$. Although many algorithms have been developed to provide conditions for an uncertain matrix to be robustly $\mathscr{D}$-stable, the problem of computing the probability of an uncertain matrix to be $\mathscr{D}$-stable is still unexplored. The goal of this paper is to fill this gap in two directions. First, the only constraint on the stability region $\mathscr{D}$ that we impose is that its complement is a semialgebraic set. This comprises many important cases in robust control theory. Second, the $\mathscr{D}$-stability analysis problem is formulated in a probabilistic framework, by assuming that only few probabilistic information is available on the uncertain parameters, such as support and some moments. We will show how to compute the minimum probability that the matrix is $\mathscr{D}$-stable by using convex relaxations based on the theory of moments.**

***Index Terms*—Robust and Probabilistic $\mathscr{D}$-stability, Uncertain polynomial matrices, Theory of moments.**

I. Introduction

In control theory, robust stability and performance requirements can be formulated in terms of robust $\mathscr{D}$-stability analysis, which aims at verifying if all the eigenvalues of an uncertain matrix lie in a given region $\mathscr{D}$ of the complex plane. A control law that guarantees robust performance is in general very conservative. A less conservative controller may be obtained by guaranteeing the performance probabilistically, e.g., we may require the performance to be achieved within a given level of probability. This problem can be formulated as probabilistic $\mathscr{D}$-stability analysis, whose aim is to compute the probability that all the eigenvalues of an uncertain matrix lie in $\mathscr{D}$. In this work, we present a unified framework to assess robust and probabilistic $\mathscr{D}$-stability of uncertain matrices. Specifically, the contribution is twofold.

First, a novel approach for analysing robust $\mathscr{D}$-stability of an uncertain matrix $A(\rho)$ is proposed. The entries of $A(\rho)$ depend polynomially on an uncertain parameter vector ρ, which takes values in a closed semialgebraic set Δ described by polynomial constraints. The only assumption on the stability region $\mathscr{D}$ is that its complement is a semialgebraic set, described by polynomial constraints in the complex plane. The addressed problem is quite general and it includes, among others, the analysis of robust nonsingularity, Hurwitz or Schur stability of matrices with polytopic or 2-norm bounded perturbations.

Second, the $\mathscr{D}$-stability analysis problem is formulated in a probabilistic framework, by assuming that the uncertain parameters ρ are described by a set of non a-priori specified probability measures. Only the support and some moments of the probability measures characterizing the uncertainty ρ are assumed to be known. This is an approach to robustness, based on *Imprecise Probability* [1], that has been recently developed in filtering theory [2]–[4]. Specifically, we seek the “worst-case probabilistic scenario”, which requires to compute, among all possible probability measures satisfying the assumptions, the smallest probability of the matrix $A(\rho)$ to be $\mathscr{D}$-stable. This allows us to take into account not only the information about the range of the parameter ρ (i.e., $\rho \in \Delta$), but also information such as its nominal value, its variance and so on. This information allows us to reduce the conservativeness of the obtained results, at the price of guaranteeing $\mathscr{D}$-stability within a given level of probability.

To this end, we develop a unified framework for deterministic (robust) and probabilistic $\mathscr{D}$-stability analysis. A semi-infinite linear program is formulated and then relaxed, by exploiting the Lasserre's hierarchy [5], into a sequence of *semidefinite programming* (SDP) problems of finite size.

<u>Related works:</u> Evaluating the properties of the eigenvalues of a family of matrices (e.g., spectral radius) is an NP-hard problem [6], and there is a vast literature on this topic [7]–[22]. This list is far from being exhaustive, but it points out the efforts made to develop methodologies that, in many cases, can be applied to tackle specific robust $\mathscr{D}$-stability analysis problems (e.g., Hurwitz or Schur stability) under specific assumptions on the uncertainty (e.g., polytopic or 2-norm bounded uncertainty). In the context of the present paper, we mention [23] and [24], where Lasserre's hierarchy is used to approximate the stability region of univariate polynomials with uncertain coefficients. In Section V-A, we compare these works with ours.

To the best of our knowledge, there are not previous works on probabilistic $\mathscr{D}$-stability analysis.

<u>Notation:</u> Let us denote with x_{re} and x_{im} the real and imaginary part, respectively, of a complex vector x. Let z_i be the i-th component of a vector $z \in \mathbb{R}^{n_z}$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}_0^{n_z}$ the set of n_z-dimensional vectors with non-negative integer components. For an integer τ, $\mathscr{A}_\tau^{n_z}$ is the set

* The authors are with the IDSIA Dalle Molle Institute for Artificial Intelligence SUPSI-USI, 6928 Manno, Switzerland. e-mails: dario@idsia.ch; alessio@idsia.ch

This work was partially supported by the European Commission under project H2020-723248 “DAEDALUS - Distributed control and simulation platform to support an ecosystem of digital automation developers”.

defined as $\left\{\alpha \in \mathbb{N}_0^{n_z} : \sum_{i=1}^{n_z} \alpha_i \leq \tau\right\}$. We will use the shorthand notation z^α for $z^\alpha = z_1^{\alpha_1} \cdots z_{n_z}^{\alpha_{n_z}} = \prod_{i=1}^{n_z} z_i^{\alpha_i}$. Let us denote with $\mathbb{R}_\tau[z]$ the set of real-valued polynomials in $z \in \mathbb{R}^{n_z}$ with degree less than or equal to τ, and let $b_\tau(z)$ be the canonical basis of $\mathbb{R}_\tau[z]$, i.e., $b_\tau(z) = \{z^\alpha\}_{\alpha \in \mathscr{A}_\tau^{n_z}}$. Denote with $\{\boldsymbol{g}_\alpha\}_{\alpha \in \mathscr{A}_\tau^{n_z}}$ the coefficients of the polynomial $g \in \mathbb{R}_\tau[z]$ in the canonical basis $b_\tau(z)$, i.e., $g(z) = \sum_{\alpha \in \mathscr{A}_\tau^{n_z}} \boldsymbol{g}_\alpha z^\alpha$. In the case g is an n_g-dimensional vector of polynomials in $\mathbb{R}_\tau[z]$, we denote with $\boldsymbol{g}_{i,\alpha}$ the coefficients of the polynomial g_i in the basis $b_\tau(z)$. Let us denote with $deg(g)$ the degree of the polynomial g.

Let P_z be the cumulative distribution function of a Borel probability measure Pr_z on $\mathbb{R}^{n_z}$. Because of the equivalence between Borel probability measures and cumulative distributions, we will use interchangeably Pr_z and P_z.

II. Problem setting

A. Uncertainty description

Consider an uncertain square real matrix $A(\rho)$ of size n_a, whose entries depend polynomially on an uncertain parameter vector $\rho \in \mathbb{R}^{n_\rho}$. The uncertain vector ρ is assumed to belong to a compact semialgebraic uncertainty set Δ, defined as

$$\Delta = \left\{\rho \in \mathbb{R}^{n_\rho} : g_i(\rho) \geq 0, \quad i = 1, \ldots, n_g\right\}, \tag{1}$$

where g_i are real-valued polynomial functions of ρ.

We also assume to have some probabilistic information on the uncertain vector ρ. Specifically, given n_f real-valued polynomial functions f_i $(i = 1, \ldots, n_f)$ called *generalized polynomial moment functions* (gpmfs) and defined on Δ, we assume that the probabilistic information on the vector ρ is represented by the expectations of the gpmfs f_i, i.e.,

$$\mathbb{E}[f_i] = \int_\Delta f_i(\rho) dP_\rho(\rho) = \mu_i, \quad i = 1, \ldots, n_f, \tag{2}$$

where the integral is a Lebesgue-Stieltjes integral with respect to the cumulative distribution function P_ρ of a Borel probability measure Pr_ρ on Δ[1] and $\mu_i \in \mathbb{R}$ are finite and known.[2] We will assume $f_1(\rho) = 1$ and $\mu_1 = 1$, which expresses the fact that Pr_ρ is a probability measure with support on Δ, i.e.,

$$\mathbb{E}[f_1] = \int_\Delta dP_\rho(\rho) = 1 = \mathrm{Pr}_\rho(\rho \in \Delta).$$

Note that the knowledge of the expectation of n_f gpmfs f_i is not enough to uniquely define the measures of probability Pr_ρ, thus we consider the set of all probability measures Pr_ρ which are compatible with the information in (2):

$$\mathscr{P}_\rho = \left\{P_\rho : \int_\Delta f_i(\rho) dP_\rho(\rho) = \mu_i, \quad i = 1, \ldots, n_f\right\}. \tag{3}$$

With some abuse of terminology, when in the rest of the paper we state that the probability measures Pr_ρ belong to $\mathscr{P}_\rho$, we actually mean that the corresponding cumulative distribution functions P_ρ belong to $\mathscr{P}_\rho$.

[1] The sample space is $\mathbb{R}^{n_\rho}$ and we are considering the Borel σ-algebra. Δ is assumed to be an element of the σ-algebra.

[2] Although equality constraints on the gpmfs f_i are considered in (2), inequality constraints can be also handled.

***Example** 1.1:* Let us introduce a simple example which will be used throughout the paper for illustrative purposes. Let the uncertain matrix be

$$A(\rho) = \begin{bmatrix} \rho - 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{4}$$

with $\rho \in \Delta = [0, 1]$. According to the notation in (1), the set Δ is written as $\Delta = \{\rho \in \mathbb{R} : g_1(\rho) \doteq \rho \geq 0, \quad g_2(\rho) \doteq 1 - \rho \geq 0\}$. Two cases are considered.

First, the information about ρ is expressed by the set of probability measures:

$$\mathscr{P}_\rho^{(1)} = \left\{P_\rho : \int_0^1 dP_\rho(\rho) = 1\right\}. \tag{5}$$

This means that only the support $\Delta = [0, 1]$ of the probability measures P_ρ is known.

Second, the information about ρ is expressed by:

$$\mathscr{P}_\rho^{(2)} = \left\{P_\rho : \int_0^1 dP_\rho(\rho) = 1, \ \int_0^1 f_2(\rho)\, dP_\rho(\rho) = 0.5\right\}, \tag{6}$$

with $f_2(\rho) = \rho$. This means that both the support and the first moment of the probability measures Pr_ρ are known (i.e., the mean is assumed to be 0.5). ■

The problems reported in the next paragraphs are addressed.

B. Probabilistic $\mathscr{D}$-stability analysis

A matrix is $\mathscr{D}$-stable if all the eigenvalues belong to a given region $\mathscr{D}$. In this paper, we assume that $\mathscr{D}$ is an open subset of the complex plane, whose complement $\mathscr{D}^c = \mathbb{C} \setminus \mathscr{D}$ (instability region) is a closed semialgebraic set described by

$$\mathscr{D}^c = \{\lambda \in \mathbb{C} : d_i(\lambda_{\mathrm{re}}, \lambda_{\mathrm{im}}) \geq 0, \ i = 1, \ldots, n_d\}, \tag{7}$$

with d_i being real-valued polynomials in the real variables λ_{re} and λ_{im}. Note that $\mathscr{D}$ can be, for instance, the open left half plane, the unitary disk centered in the origin, or the complex plane without the imaginary axis. Therefore, this assumption cover many important cases in stability analysis.

Among all the probability measures belonging to $\mathscr{P}_\rho$, we want to find the "worst-case scenario" given by the measure of probability Pr_ρ which provides the lower probability that $A(\rho)$ has all the eigenvalues in $\mathscr{D}$ or, equivalently, the upper probability $\overline{p} = 1 - \underline{p}$ that $A(\rho)$ has at least an eigenvalue in $\mathscr{D}^c$. In this way, we can claim that the probability of the matrix $A(\rho)$ to be $\mathscr{D}$-stable w.r.t. the uncertainties ρ is greater than or equal to $\underline{p}$ (equiv. $1 - \overline{p}$). Formally, we are interested in solving the following eigenvalue location problem.

***Problem** 1:* **[Probabilistic eigenvalue violation]** Given the uncertain matrix $A(\rho)$, the uncertain parameter vector ρ with (unknown) measure of probability Pr_ρ belonging to $\mathscr{P}_\rho$, and a stability region $\mathscr{D}$, compute

$$\overline{p} = \sup_{P_\rho \in \mathscr{P}_\rho} \mathrm{Pr}_\rho\left(\Lambda(A(\rho)) \nsubseteq \mathscr{D}\right), \tag{8}$$

where $\Lambda(A(\rho))$ is the spectrum of $A(\rho)$. ■

***Example** 1.2:* Let us consider the running example. As a stability region, we consider the open left half-plane $\mathscr{D} = \{\lambda \in \mathbb{C} \mid \lambda_{re} < 0\}$, whose complement is the semi-algebraic set: $\mathscr{D}^c = \{\lambda \in \mathbb{C} \mid d_1(\lambda_{re}) \doteq \lambda_{re} \geq 0\}$. Since the eigenvalues of $A(\rho)$ in (4) are -1 and $\rho-1$, the only eigenvalue that can lead to instability is $\rho-1$. Therefore, problem (8) becomes:

$$\overline{p} = \sup_{P_\rho \in \mathscr{P}_\rho} \Pr_\rho(\rho - 1 \geq 0), \tag{9}$$

where we have exploited that $\mathscr{D}^c = \{\lambda \in \mathbb{C} \mid \lambda_{re} \geq 0\}$. ■

The following theorem shows that the challenging problem of verifying deterministic (robust) $\mathscr{D}$-stability of $A(\rho)$ is a special case of Problem 1.

***Theorem** 1 (Deterministic eigenvalue violation):* If the set $\mathscr{P}_\rho$ is described by $\mathscr{P}_\rho = \left\{P_\rho : \int_\Delta dP_\rho(\rho) = 1\right\}$, then the solution $\overline{p}$ of problem (8) is

$$\overline{p} = \begin{cases} 1 & \text{if } A(\rho) \text{ is not robustly } \mathscr{D}\text{-stable}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: First of all observe that

$$\Pr_\rho(\Lambda(A(\rho)) \nsubseteq \mathscr{D}) = \int_\Delta (1 - \mathbb{I}_\mathscr{D}(\Lambda(A(\rho))))\, dP_\rho(\rho),$$

where $\mathbb{I}_\mathscr{D}(\Lambda(A(\rho)))$ is the indicator function:

$$\mathbb{I}_\mathscr{D}(\Lambda(A(\rho))) = \begin{cases} 1 & \text{if } \Lambda(A(\rho)) \subseteq \mathscr{D}, \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

and $P_\rho \in \mathscr{P}_\rho$ with $\mathscr{P}_\rho = \left\{P_\rho : \int_\Delta dP_\rho(\rho) = 1\right\}$. $\mathscr{P}_\rho$ includes all the probability measures supported by Δ and so it also includes atomic measures (Dirac's delta) with support in Δ. Hence, assume that the matrix $A(\rho)$ is not robustly $\mathscr{D}$-stable against Δ. Thus, there exists $\hat{\rho} \in \Delta$ such that $\Lambda(A(\hat{\rho})) \nsubseteq \mathscr{D}$. Then we can take $\Pr_\rho$ equal to the Dirac's delta centred on $\hat{\rho}$ and we have that $\Pr_\rho(\Lambda(A(\hat{\rho})) \nsubseteq \mathscr{D}) = 1$. Similarly, if $A(\rho)$ is $\mathscr{D}$-stable for any $\rho \in \Delta$. Then $(1 - \mathbb{I}_\mathscr{D}(\Lambda(A(\rho)))) = 0$ for any $\rho \in \Delta$. Thus, $\Pr_\rho(\Lambda(A(\hat{\rho})) \nsubseteq \mathscr{D}) = 0$. ■

The assumption in Theorem 1 means that the only information on ρ is the support Δ of the probability measures $\Pr_\rho$ (namely, we only know that $\rho \in \Delta$). The theorem shows that the deterministic $\mathscr{D}$-stability analysis problem is a particular case of probabilistic $\mathscr{D}$-stability analysis. Although Theorem 1 is quite intuitive, it is fundamental to formulate, in a rigorous way, the deterministic and the probabilistic $\mathscr{D}$-stability analysis problem in a unified framework. In fact, one could erroneously think that the probabilistic constraint equivalent to $\rho \in \Delta$ is

$$\int_\Delta \frac{1}{|\Delta|} d\rho = 1, \tag{11}$$

where $|\Delta|$ is the Lebesgue measure of Δ, i.e., $\Pr_\rho$ is equal to the uniform distribution on Δ. This is not correct, as illustrated in the following example.

***Example** 1.3:* If in the running example we translate the (deterministic) information $\rho \in \Delta$ as in (11) (with $|\Delta| = 1$), the probability that the matrix is unstable would be equal to zero, since the only value that gives instability ($\rho = 1$) has zero measure. The mistake here is that the uniform distribution is just one of the possible probability measures with support on Δ. There are infinite of such distributions and, as discussed above, the one that gives rise to instability is an atomic measure on $\rho = 1$. Thus, the equivalent of the constraint $\rho \in \Delta$ is $P_\rho \in \mathscr{P}_\rho^{(1)} = \left\{P_\rho : \int_0^1 dP_\rho(\rho) = 1\right\}$ and not (11). ■

III. A MOMENT PROBLEM FOR $\mathscr{D}$-STABILITY ANALYSIS

As shown in Theorem 1, the problem of evaluating robust $\mathscr{D}$-stability of an uncertain matrix $A(\rho)$ is a particular case of probabilistic $\mathscr{D}$-stability analysis. However, for the sake of exposition, we first provide results in the deterministic setting. The probabilistic scenario will be discussed later.

A. Checking determinist $\mathscr{D}$-stability

The following theorem (based on an extension of the results in [25]) provides necessary and sufficient conditions to check determinist $\mathscr{D}$-stability of the matrix $A(\rho)$ against the uncertainty set Δ.

***Theorem** 2:* All the eigenvalues of $A(\rho)$ are located in the set $\mathscr{D}$ for all uncertainties $\rho \in \Delta$ if and only if the solution of the following (nonconvex) optimization problem is 0:

$$\max_{x \in \mathbb{C}^{n_a}, \rho \in \Delta, \lambda \in \mathbb{C}} \quad \|x\|_2^2 \tag{12a}$$

$$s.t. \quad (A(\rho) - \lambda I)x = 0, \quad \|x\|^2 \leq 1, \quad \lambda \in \mathscr{D}^c. \tag{12b}$$

Proof: First, the "only if" part is proven. If all the eigenvalues of $A(\rho)$ are located in $\mathscr{D}$, there exists no $\lambda \in \mathscr{D}^c$ and $\rho \in \Delta$ which make the matrix $A(\rho) - \lambda I$ singular. Thus, only the trivial solution $x = 0$ satisfies the constraint $(A(\rho) - \lambda I)x = 0$. Therefore, the solution of problem (12) is equal to zero. The "if" part is proven by contradiction. Assume there exists an uncertainty $\rho \in \Delta$ such that an eigenvalue λ_i of $A(\rho)$ belongs to $\mathscr{D}^c$. Thus, the corresponding eigenvector $x^* \neq 0$ satisfies the constraint $(A(\rho) - \lambda_i I)x^* = 0$. Furthermore, for any $\beta \in \mathbb{C}$, also $x = \beta x^*$ satisfies the constraint $(A(\rho) - \lambda_i I)x = 0$. Thus, the supremum of the 2 norm of the set of vectors x satisfying $(A(\rho) - \lambda_i I)x = 0$ is infinity. Because of the constraint $\|x\|^2 \leq 1$, the solution of problem (12) is 1, contradicting the hypothesis. ■

Corollary 1: There exists an uncertainty $\rho \in \Delta$ such that at least an eigenvalue λ_i of $A(\rho)$ does not belong to $\mathscr{D}$ if and only if the solution of problem (12) is 1. ■

B. Checking probabilistic $\mathscr{D}$-stability

Let us now focus on the probabilistic $\mathscr{D}$-stability analysis problem, which aims at computing $\overline{p}$, namely, the upper probability among the probability measures in $\mathscr{P}_\rho(\mu)$ of the matrix $A(\rho)$ to have at least an eigenvalue in the instability region $\mathscr{D}^c$ (see Problem 1). The following theorem, which can be seen as the probabilistic version of Theorem 2 and Corollary 1, shows how the computation $\overline{p}$ can be formulated as a moment optimization problem.

***Theorem** 3:* Given the uncertain matrix $A(\rho)$, the uncertain parameter vector ρ whose measures of probability $\Pr_\rho(\rho)$ are constraint to belong to $\mathscr{P}_\rho$, and the instability region $\mathscr{D}^c$, the upper probability $\overline{p}$ (defined in (8)) of the matrix $A(\rho)$ to have at least an eigenvalue in $\mathscr{D}^c$ is given by the solution of the following optimization problem:

$$\overline{p} = \sup_{P_{\rho,x,\lambda}} \iiint \|x\|^2 dP_{\rho,x,\lambda}(\rho,x,\lambda) \tag{13a}$$

$$s.t. \iiint dP_{\rho,x,\lambda}(\rho,x,\lambda) = 1, \tag{13b}$$

$$\int_{\rho\in\Delta}\int_{\|x\|^2\leq 1}\int_{\lambda\in\mathscr{D}^c} dP_{\rho,x,\lambda}(\rho,x,\lambda) = 1, \tag{13c}$$

$$\int_{\rho\in\Delta}\int_{\|x\|^2\leq 1}\int_{\lambda\in\mathscr{D}^c} f_i(\rho) dP_{\rho,x,\lambda}(\rho,x,\lambda) = \mu_i,\ i=2,\ldots,n_f, \tag{13d}$$

$$\iiint_{(A(\rho)-\lambda I)x=0} dP_{\rho,x,\lambda}(\rho,x,\lambda) = 1, \tag{13e}$$

with $P_{\rho,x,\lambda}$ being the joint cumulative distribution function of the variables (ρ,x,λ). ■

A detailed proof of Theorem 3 can be found in [26, Sec. IV.B]. The intuitive explanation of Theorem 3 is the following. According to Theorem 2 and Corollary 1, when the optimum of the deterministic problem (12) is achieved, $\|x\|^2 = 1$ if $\Lambda(A(\rho)) \nsubseteq \mathscr{D}$, 0 otherwise. Thus, when the information on ρ is modeled in terms of probability measures, $\|x\|^2$ becomes a uncertain variable which takes value 0 if $\Lambda(A(\rho)) \subseteq \mathscr{D}$, 1 if $\Lambda(A(\rho)) \nsubseteq \mathscr{D}$. Thus, the expected value of $\|x\|^2$ (namely, the objective function in (13)) coincides with $\Pr_x(\|x\|^2 = 1)$, which in turn provides $\Pr_\rho(\Lambda(A(\rho)) \nsubseteq \mathscr{D})$.

The constraints in (13c) and (13e) are simply the "probabilistic version" of the determinist constraints in (12), and they are used to describe the support of the probability measures $\Pr_{\rho,x,\lambda}$. The constraint (13d) includes the information in (2) on the (generalized) moments of the probability measures $\Pr_\rho$, i.e., $P_\rho \in \mathscr{P}_\rho$.

Example 1.4: Let us continue the explanatory example, and consider the case where the probabilistic information on ρ is expressed by the set $\mathscr{P}_\rho^{(2)}$ (eq. (6)). Then, problem (13) is given by:

$$\overline{p} = \sup_{P_{\rho,x,\lambda}} \iiint \|x\|^2 dP_{\rho,x,\lambda}(\rho,x,\lambda) \tag{14a}$$

$$s.t. \int dP_{\rho,x,\lambda}(\rho,x,\lambda) = 1, \tag{14b}$$

$$\int_{\rho\in[0\ 1]}\int_{\|x\|^2\leq 1}\int_{\lambda_{\text{re}}\geq 0} dP_{\rho,x,\lambda}(\rho,x,\lambda) = 1, \tag{14c}$$

$$\int_{\rho\in[0\ 1]}\int_{\|x\|^2\leq 1}\int_{\lambda_{\text{re}}\geq 0} \rho\, dP_{\rho,x,\lambda}(\rho,x,\lambda) = 0.5, \tag{14d}$$

$$\iiint_{(A(\rho)-\lambda_{\text{re}} I)x=0} dP_{\rho,x,\lambda}(\rho,x,\lambda) = 1. \tag{14e}$$

Because of the constraint (14c), the joint distribution $\Pr_{\rho,x,\lambda}$ is supported by

$$\left\{(\rho,x,\lambda_{\text{re}}):\quad \rho\in[0\ 1],\ \|x\|^2\leq 1,\ \lambda_{\text{re}}\geq 0\right\}.$$

We remind that $A(\rho)$ is unstable if and only if $\rho = 1$. For this value of ρ, $A(\rho)$ has an eigenvalue in zero. Let us rewrite $P_{\rho,x,\lambda}$ as $P_x(\cdot|\rho,\lambda_{\text{re}})P_{\rho,\lambda_{\text{re}}}$. Then, because of (14e), the conditional marginal distribution $P_x(\cdot|\rho,\lambda_{\text{re}})$ is supported by:

$$\left\{x\colon \|x_1\|^2 \leq 1,\ x_2 = 0\right\} \text{ if } \rho = 1 \text{ and } \lambda_{\text{re}} = 0,$$
$$\{0\} \text{ if } \rho \neq 1 \text{ or } \lambda_{\text{re}} \neq 0.$$

Thus, at the optimum, the objective function of problem (14) is given by

$$\int_{\rho=1}\int_{\lambda_{\text{re}}=0} dP_{\rho,\lambda}(\rho,\lambda). \tag{15}$$

Among all the probability measures $\Pr_{\rho,\lambda}$ satisfying the moment constraint (14d) on the marginal distribution $\Pr_\rho$ and the constraints (14c)-(14e), the one maximizing (15) is given by

$$\Pr_{\rho,\lambda}(\rho,\lambda) = \left(0.5\delta_{(0)}(\rho) + 0.5\delta_{(1)}(\rho)\right)\delta_{(0)}(\lambda_{\text{re}}). \tag{16}$$

Thus, the maximum value of (15) is given by:

$$\int_{\rho=1}\int_{\lambda_{\text{re}}=0} dP_{\rho,\lambda}(\rho,\lambda) =$$
$$\int_{\rho=1}\left(0.5\delta_{(0)}(\rho) + 0.5\delta_{(1)}(\rho)\right) d\rho \int_{\lambda_{\text{re}}=0} \delta_{(0)}(\lambda_{\text{re}}) d\lambda_{\text{re}} = 0.5.$$

Therefore, by exploiting the information on the mean we can reduce the upper probability of instability from 1 to 0.5. ■

IV. SOLVING MOMENT PROBLEMS THROUGH SDP RELAXATIONS

Note that, in problem (13): (i) the decision variables are the amount of non-negative mass $\Pr_{\rho,x,\lambda}$ assigned to each point (ρ,x,λ), (ii) the objective function and the constraints are linear in the optimization variables $P_{\rho,x,\lambda}$. Therefore, (13) is a *semi-infinite linear program*, with a finite number of constraints but with infinite number of decision variables. In this section, we discuss how to relax (13) into a hierarchy of *semidefinite programming* (SDP) problems of finite dimension.

Let us first introduce the augmented variable vector $z = \begin{bmatrix}x_{\text{re}}^\top & x_{\text{im}}^\top & \rho^\top & \lambda_{\text{re}} & \lambda_{\text{im}}\end{bmatrix}^\top \in \mathbb{R}^{n_z}$ (with $n_z = 2n_a + n_\rho + 2$) and, with some abuse of notation, let us define $h(z) = \|x\|^2$ and $\tilde{f}(z) = f(\rho)$. Problem (13) can be then rewritten in terms of the augmented variable z and the cumulative distribution function P_z as

$$\overline{p} = \sup_{P_z} \int h(z) dP_z(z) \tag{17a}$$

$$s.t. \int dP_z(z) = 1, \quad \int \tilde{f}_i(z) dP_z(z) = \mu_i,\ i=2,\ldots,n_f, \tag{17b}$$

$$\int_{\boldsymbol{Z}} dP_z(z) = 1, \tag{17c}$$

where $\boldsymbol{Z}$ defines the support of the probability measure $\Pr_z$, and it is written in the compact form

$$\boldsymbol{Z} = \{z\in\mathbb{R}^{n_z}:\ q_j(z)\geq 0,\quad j=1,\ldots,n_q\}, \tag{18}$$

with $q_j(z)$ being real-valued polynomial functions in z, properly defined based on the polynomial constraints defining the integral domain in (13).

Example 1.5: Since in the explanatory example considered so far $A(\rho)$ is a real symmetric matrix, its eigenvalues are real, and thus we considered an augmented variable vector z:

$$z = [\rho\ \ \lambda_{\text{re}}\ \ x_1\ \ x_2]^\top \in \mathbb{R}^4. \tag{19}$$

The objective function $h(z)$ is $h(z) = z_3^2 + z_4^2$, and the components of the vector-valued function $\tilde{f}(z)$ defining the constraints on the moments is $\tilde{f}_1(z) = 1$ and $\tilde{f}_2(z) = z_1$. According to the integral domain in problem (13), the set $\boldsymbol{Z}$ defining the support of the probability measure $\Pr_z$ is given by:

$$\begin{aligned}\boldsymbol{Z} = \Big\{ & z = [\rho \;\; \lambda_{\text{re}} \;\; x_1 \;\; x_2]^\top : \\ & q_1(z) \doteq z_1 \geq 0, \quad q_2(z) \doteq 1 - z_1 \geq 0, \quad q_3(z) \doteq z_2 \geq 0, \\ & q_4(z) \doteq (z_1 - 1) z_3 \geq 0, \; q_5(z) \doteq -(z_1 - 1) z_3 \geq 0, \\ & q_6(z) \doteq z_4 \geq 0, \; q_7(z) \doteq -z_4 \geq 0, \; q_8(z) \doteq 1 - z_3^2 - z_4^2 \geq 0 \Big\}\end{aligned}$$

■

For an integer $\tau \in \mathbb{N}: \;\tau \geq \tilde{\tau}$, with

$$\tilde{\tau} = \max \left\{ 1, \max_{i=1,\ldots,n_{\tilde{f}}} \left\lceil \frac{deg(\tilde{f}_i)}{2} \right\rceil, \max_{j=1,\ldots,n_q} \left\lceil \frac{deg(q_j)}{2} \right\rceil \right\}, \quad (20)$$

let us rewrite $h(z) \in \mathbb{R}_{2\tau}[z]$ and each component $\tilde{f}_i(z) \in \mathbb{R}_{2\tau}[z]$ of the vector-valued function $\tilde{f}(z)$ as

$$h(z) = \sum_{\alpha \in \mathscr{A}_{2\tau}^{n_z}} \boldsymbol{h}_\alpha z^\alpha, \qquad \tilde{f}_i(z) = \sum_{\alpha \in \mathscr{A}_{2\tau}^{n_z}} \tilde{\boldsymbol{f}}_{i,\alpha} z^\alpha, \qquad (21)$$

where, according to the notation introduced in Section I, $\boldsymbol{h}_\alpha$ (resp. $\tilde{\boldsymbol{f}}_{i,\alpha}$) are the coefficients of the polynomial $h(z)$ (resp. $\tilde{f}_i(z)$). Based on eq. (21), we can write

$$\int h(z) dP_z(z) = \int \left(\sum_{\alpha \in \mathscr{A}_{2\tau}^{n_z}} \boldsymbol{h}_\alpha z^\alpha \right) dP_z(z) = \sum_{\alpha \in \mathscr{A}_{2\tau}^{n_z}} \boldsymbol{h}_\alpha m_\alpha,$$

where m_α are the moments of the probability measure $\Pr_z$, i.e., $m_\alpha = \int z^\alpha dP_z(z)$. Similar considerations hold for the polynomial $\tilde{\boldsymbol{f}}_i(z)$.

Thus, solving problem (17) is equivalent to solve:

$$\begin{aligned}\overline{p} = & \sup_{m=\{m_\alpha\}_{\alpha \in \mathscr{A}_{2\tau}^{n_z}}} \sum_{\alpha \in \mathscr{A}_{2\tau}^{n_z}} \boldsymbol{h}_\alpha m_\alpha \qquad (22) \\ & s.t. \sum_{\alpha \in \mathscr{A}_{2\tau}^{n_z}} \tilde{\boldsymbol{f}}_{i,\alpha} m_\alpha = \mu_i, \quad i = 2, \ldots, n_f, \\ & \quad m \text{ is a sequence of moments generated by a} \\ & \quad \text{probability measure with support on } \boldsymbol{Z}.\end{aligned}$$

Comparing (17) and (22) is evident that now the optimization variables are the moments m_α (real numbers), where the constraint "$\Pr_z$ is a probability measure on $\boldsymbol{Z}$" has been replaced by "m is a sequence of moments generated by a probability measure with support on $\boldsymbol{Z}$".

The following LMIs provide necessary conditions for the sequence $m = \{m_\alpha\}_{\alpha \in \mathscr{A}_{2\tau}^{n_z}}$ to be a sequence of moments generated by a probability measure $\Pr_z(z)$ with support on $\boldsymbol{Z}$:

$$m_{0\cdots0} = 1, \; M_\tau(m) \succeq 0, \quad M_{\tau - \left\lceil \frac{deg(q_j)}{2} \right\rceil}(q_j m) \succeq 0, \quad j = 1, \ldots n_q, \qquad (23)$$

where $M_\tau(m)$ is the so-called *truncated moment matrix* of order τ and $M_{\tau - \left\lceil \frac{deg(q_j)}{2} \right\rceil}(q_j m)$ is the so-called *truncated localizing matrix* associated to the polynomial q_j. The reader is referred to [5] and [26, Sec. V] for definition of moment and localizing matrices.

According to Lasserre's hierarchy [5], instead of requiring the conditions in (22), one may require the weaker conditions (23), thus relaxing problem (22) into the so called SDP-relaxed problem of order τ:

$$\bar{p}^\tau = \sup_{m=\{m_\alpha\}_{\alpha \in \mathscr{A}_{2\tau}^{n_z}}} \sum_{\alpha \in \mathscr{A}_{2\tau}^{n_z}} \boldsymbol{h}_\alpha m_\alpha \qquad (24a)$$

$$s.t. \sum_{\alpha \in \mathscr{A}_{2\tau}^{n_z}} \tilde{\boldsymbol{f}}_{i,\alpha} m_\alpha = \mu_i, \quad i = 2, \ldots, n_f, \qquad (24b)$$

$$m_{0\cdots0} = 1, \quad M_\tau(m) \succeq 0, \qquad (24c)$$

$$M_{\tau - \left\lceil \frac{deg(q_j)}{2} \right\rceil}(q_j m) \succeq 0, \quad j = 1, \ldots n_q. \qquad (24d)$$

Since the constraints in (24) are less restrictive than the constraints in (22), $\bar{p}^\tau$ is an upper bound of $\bar{p}$. Furthermore, the moment and the localizing matrices are such that:

$$M_{\tau+1}(m) \succeq 0 \Rightarrow M_\tau(m) \succeq 0,$$
$$M_{\tau+1-\left\lceil \frac{deg(q_j)}{2} \right\rceil}(q_j m) \succeq 0 \Rightarrow M_{\tau-\left\lceil \frac{deg(q_j)}{2} \right\rceil}(q_j m) \succeq 0.$$

This implies: $\bar{p}^\tau \geq \bar{p}^{\tau+1} \geq \bar{p}$. Thus, as the relaxation order τ increases, the SDP relaxation (24) becomes tighter. Furthermore, under mild restrictive assumptions on the description of $\boldsymbol{Z}$, the solution of the relaxed problem (24) converges to the global optimum $\bar{p}$ of the original problem (17), i.e.,

$$\lim_{\tau \to \infty} \bar{p}^\tau = \bar{p}. \qquad (25)$$

Convergence in (25) follows from convergence properties of Lasserre's hierarchy. A detailed proof of (25) is reported in [26, Appendix].

***Remark** 1:* The number N_τ of the optimization variables $m = \{m_\alpha\}_{\alpha \in \mathscr{A}_{2\tau}^{n_z}}$ in (24) is given by the binomial expression: $N_\tau = \begin{pmatrix} n_z + 2\tau \\ 2\tau \end{pmatrix} = O\left(n_z^{2\tau}\right)$, and thus, for fixed relaxation order τ, N_τ grows polynomially with the size of z. ■

***Property** 1:* Since the relaxed SDP problem (24) provides an upper bound of $\bar{p}$ (i.e., $\bar{p}^\tau \geq \bar{p}$), sufficient conditions on the $\mathscr{D}$-stability of $A(\rho)$ can be derived from $\bar{p}^\tau$. Specifically, if the only information on the uncertain parameter ρ is the support Δ of its probability measures (i.e., $\rho \in \Delta$), then, from Theorem 2 and Corollary 1, $\bar{p}$ can be either 0 ($A(\rho)$ is robustly $\mathscr{D}$-stable) or 1 ($A(\rho)$ is not robustly $\mathscr{D}$-stable). Thus, if $\bar{p}^\tau < 1$, we can claim that $\bar{p} = 0$ and thus $A(\rho)$ is guaranteed to be robustly $\mathscr{D}$-stable against the uncertainty set Δ. On the other hand, if $\bar{p}^\tau \geq 1$, no conclusions can be drawn on $\mathscr{D}$-stability of $A(\rho)$. Conversely, if the moments of ρ are given, then $\bar{p}$ represents the probability of $A(\rho)$ to have at least an eigenvalue in $\mathscr{D}^c$. Thus, since $\bar{p}^\tau \geq \bar{p}$, we can claim that $A(\rho)$ is not $\mathscr{D}$-stable with probability less than or equal to $\bar{p}^\tau$. Equivalently, $A(\rho)$ is $\mathscr{D}$-stable with probability at least $1 - \bar{p}^\tau$. ■

***Example** 1.6:* Let us go back to the explanatory example. For a relaxation order $\tau = 2$, the solution of the SDP problem (24) is $\overline{p}^\tau = 0.5$. Thus, we can claim that $A(\rho)$ is not $\mathscr{D}$-stable with probability at most $\overline{p}^\tau = 0.5$. Note that $\overline{p}^\tau$ is a tight solution (i.e., $\overline{p}^\tau = \overline{p}$). In fact, we have seen in Example 1.4 that, for a probability measure $\Pr_\rho = 0.5\delta_{(0)} + 0.5\delta_{(1)}$, the matrix $A(\rho)$ has an eigenvalue equal to 0 with probability 0.5.

V. Applications and Examples

In this section, we show the application of the proposed approach through two numerical examples. The problem of robust Hurwitz stability of uncertain matrices is addressed in the first example. Robust and probabilistic analysis of the properties of uncertain dynamical models is discussed in the second examples, taken from [22], where sufficient conditions for nonexistence of bifurcations in uncertain nonlinear dynamical systems are derived. Another example on the analysis of robust stability and performance verification of LTI systems with parametric uncertainty can be found in [26, Sec. 6.3].

All computations are carried out on an i7 2.40-GHz Intel core processor with 3 GB of RAM running MATLAB R2014b. The YALMIP Matlab interface [27] is used to construct the SDP problems (24), which are solved with SeDuMi.

A. Hurwitz stability and polynomial abscissa

The aim of this example is to highlight the advantages of our approach w.r.t. the polynomial optimization based methods [23], [24]. Since [24] is focused on the approximation of the abscissa of an uncertain polynomial (i.e., maximum real part of the roots of a univariate polynomial), a robust Hurwitz stability analysis problem is discussed.

Let us consider the uncertain matrix

$$A(\rho) = \begin{bmatrix} -2.4-\rho_1^2 & 6-\rho_1^2 \\ 1-2\rho_1^2 & -2.9-2\rho_1 \end{bmatrix}, \tag{26}$$

with $\rho_1 \in \Delta = [-0.1 \quad 3.4]$, whose characteristic polynomial is:

$$\begin{aligned} P(s,\rho_1) =& s^2 + (5.3+2\rho_1+\rho_1^2)s+ \\ &+0.96+4.8\rho_1+15.9\rho_1^2+2\rho_1^3-2\rho_1^4. \end{aligned} \tag{27}$$

<ins>*Polynomial abscissa approximation [24]*</ins>

The main idea in [24] is to find a fixed-degree polynomial $\overline{P}_d(\rho_1)$ approximating, from above, the abscissa $a(\rho_1)$ of the polynomial $P(s,\rho_1)$. Specifically, among all the polynomials $\overline{P}_d(\rho_1)$ of given degree d such that $\overline{P}_d(\rho_1) \geq a(\rho_1) \quad \forall \rho_1 \in \Delta$, the one minimizing the integral

$$\int_{\rho_1 \in \Delta} \overline{P}_d(\rho_1) d\rho_1 \tag{28}$$

is sought. SDP relaxations based on *sum-of-squares* are then used to find $\overline{P}_d(\rho_1)$. If $\max_{\rho_1 \in \Delta} \overline{P}_d(\rho_1) < 0$, then all the roots of $P(s,\rho_1)$ have negative real part, thus the matrix $A(\rho)$ in (26) is robustly Hurwitz stable. Fig. 1 shows the abscissa $a(\rho_1)$ of the polynomial $P(s,\rho_1)$, along with computed upper approximating polynomial $\overline{P}_d(\rho_1)$ of degree $d=8$ in the interval $\Delta = [-0.1 \quad 3.4]$. As $\overline{P}_d(\rho_1) \geq 0$ for some values of $\rho_1 \in \Delta$, no conclusions can be drawn from $\overline{P}_d(\rho_1)$ on robust Hurwitz stability of the matrix $A(\rho)$. This conservativeness is due to the fact that the computed polynomial $\overline{P}_d(\rho_1)$ is the "best" (w.r.t. the integral (28)) upper approximation of the abscissa $a(\rho_1)$ over the whole uncertainty set Δ. On the other hand, in assessing robustly Hurwitz stability of $P(s,\rho_1)$, we are only interested in approximating the maximum of the abscissa.

The CPU time required to verify Hurwitz stability of $A(\rho)$ is 2.5 s. This includes the time required to compute the upper approximating polynomial $\overline{P}_d(\rho_1)$ as well as the time required to compute its maximum over ρ_1 through Lasserre's relaxation. In the case the polynomial $\overline{P}_d(\rho)$ is of large degree (say, $d > 10$), a large Lasserre's relaxation order may be needed to achieve a tight approximation of the maximum of $\overline{P}_d(\rho)$, thus leading to Lasserre's relaxations which might be computationally intractable.

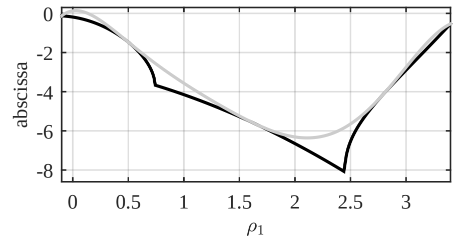

Fig. 1. Abscissa of the polynomial $P(s,\rho_1)$ in (27) (black line) and computed polynomial approximation (gray line).

<ins>*Hermite stability criterion [23]*</ins>

The problem of robust $\mathscr{D}$-stability of a polynomial is tackled in [23] approximating the minimum eigenvalue of the associated Hermite matrix. In order to check Hurwitz stability of the polynomial $P(s,\rho_1)$ in (27), the associated 2×2 symmetric Hermite matrix $H(\rho_1)$ is constructed. Since the coefficients of $P(s,\rho_1)$ are polynomials in ρ_1 of maximum degree 4, the entries of the matrix $H(\rho_1)$ are polynomials in ρ_1 of maximum degree 8. According to the Hermite stability criterion (see [23]), $P(s,\rho_1)$ is robustly Hurwitz stable if and only if

$$H(\rho_1) \succ 0, \quad \forall \rho_1 \in \Delta. \tag{29}$$

The robust minimum eigenvalue of $H(\rho_1)$ is given by

$$\lambda_{\min} = \min_{\rho \in \Delta} \min_{x \in \mathbb{R}^2 : x^\top x = 1} x^\top H(\rho_1) x. \tag{30}$$

As well known, (29) holds, or equivalently $P(s,\rho_1)$ is robustly Hurwitz stable, if and only if $\lambda_{\min} > 0$. Then, a lower bound $\underline{\lambda}_{\min}$ of $\lambda_{\min}$ is computed solving the polynomial optimization problem (30) through the Lasserre's hierarchy, for a relaxation order $\tau = 5$, which is the minimum allowed value for τ, as the objective function in (30) is a 10-degree polynomial in the augmented variable $[x \ \rho_1]$. We obtain a lower bound $\underline{\lambda}_{\min} = 6.5$, in a CPU time of 2.7 s. The obtained results allow us to claim that $H(\rho_1)$ is robustly positive definite, thus $P(s,\rho_1)$ is robustly Hurwitz stable, and no conservativeness is introduced in relaxing (30) through Lasserre's hierarchy. However, the example shows that even if the entries of $A(\rho)$ are polynomials of degree at most 2, the functional minimized in (30) is a polynomial of degree 10, which required to use a relaxation order at least equal to $\tau = 5$. As already discussed, the Lasserre's hierarchy may become computationally intractable in the case of multidimensional uncertain parameter ρ and large relaxation orders.

<ins>*Robust $\mathscr{D}$-stability analysis*</ins>

The approach proposed in this paper is now used to assess robust Hurwitz stability of the matrix $A(\rho)$. The polynomial optimization problem (12) is formulated, and solved through the Lasserre's hierarchy for a relaxation order $\tau = 3$ (namely, the SDP problem (24) is solved without using any information on the moments of ρ_1). The obtained solution of the SDP relaxed problem (24) is 10^{-9}. Thus, according to Property 1, $A(\rho)$ is robustly Hurwitz stable. The CPU time required to assess robust Hurwitz stability of $A(\rho)$ is 1.5 s. Thus, in this simple example, the proposed approach is about 1.6x faster than the methods [23] and [24]. This is due to the fact that, in the presented approach, the Lasserre's relaxation order τ can be kept "small", as the maximum degree of the polynomial constraints in (12) is 3 because of the product $A(\rho)x$.

B. Bifurcation analysis

The example discussed in this section has been recently studied in [22], where the analysis of the location of the eigenvalues of an uncertain matrix is applied to derive sufficient conditions for nonexistence of bifurcations in nonlinear dynamical systems with parametric uncertainty.

As an example, [22] considers a continuous-time predator-prey model, described by the differential equations

$$\dot{r}_1 = \gamma r_1(1-r_1) - \frac{\rho_1 r_1 r_2}{\rho_2 + r_1}, \quad \dot{r}_2 = -\rho_3 r_2 + \frac{\rho_1 r_1 r_2}{\rho_2 + r_1}, \tag{31}$$

where r_1 and r_2 are scaled population numbers, $\gamma = 0.1$ is the prey growth rate, ρ_1, ρ_2 and ρ_3 are real uncertain parameters.

A non-trivial equilibrium point for the model (31) is:

$$r_{1,\mathrm{eq}} = \frac{\rho_2\rho_3}{\rho_1 - \rho_3}, \quad r_{2,\mathrm{eq}} = \frac{\gamma\rho_2}{\rho_1-\rho_3}\left(1 - \frac{\rho_2\rho_3}{\rho_1-\rho_3}\right). \tag{32}$$

The Jacobian J of the system at the equilibrium point $(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}})$ in (32) is

$$J(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}}) = \begin{bmatrix} \gamma\frac{\rho_3}{\rho_1}\left(1-\rho_2\frac{\rho_1+\rho_3}{\rho_1-\rho_3}\right) & -\rho_3 \\ \gamma\frac{1}{\rho_1}\left(\rho_1-\rho_3-\rho_2\rho_3\right) & 0 \end{bmatrix}. \tag{33}$$

Well known results from the bifurcation theory [28] state that a sufficient condition to guarantee the existence of no local bifurcations at the equilibrium point $(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}})$ is that $J(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}})$ has no eigenvalues with zero real part.

Let us assume that ρ_1, ρ_2, and ρ_3 take values in the intervals

$$\rho_i \in [\rho_i^{\mathrm{o}} - k\Delta\rho_i \quad \rho_i^{\mathrm{o}} + k\Delta\rho_i], \quad i = 1,2,3, \tag{34}$$

where ρ_i^{o} denotes the nominal value of the parameter ρ_i, $k \in \mathbb{R}$ is a scaling factor, and $\Delta\rho_i$ characterizes the width of the uncertainty interval where ρ_i belongs to. Like in [22], we assume that the uncertainty intervals in (34) share the same width, i.e., $\Delta\rho_i = 1$ for all $i = 1,2,3$, and they are centered at the nominal values $\rho_1^{\mathrm{o}} = 9$, $\rho_2^{\mathrm{o}} = 2$ and $\rho_3^{\mathrm{o}} = 2$.

Note that the entries of the Jacobian $J(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}})$ are not polynomial functions in the uncertain parameters ρ_1, ρ_2 and ρ_3. However, by introducing the slack variables:

$$t_1 = \frac{\rho_3}{\rho_1}, \quad t_2 = \frac{1}{\rho_1 - \rho_3},$$

the entries of the matrix $J(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}})$ can be rewritten as polynomial functions in ρ_1, ρ_2, ρ_3, t_1, t_2, i.e.,

$$J(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}}) = \begin{bmatrix} \gamma t_1\left(1-\rho_2(\rho_1+\rho_3)t_2\right) & -\rho_3 \\ \gamma(1-t_1-\rho_2 t_1) & 0 \end{bmatrix},$$

where the additional polynomial constraints:

$$\rho_1 t_1 = \rho_3, \quad (\rho_1 - \rho_3)t_2 = 1, \tag{35}$$

have to be considered along with the interval constraints (34) on ρ_i to maintain the relationship among the entries of the matrix $J(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}})$. This leads to an augmented set of uncertain variables (namely, $\rho_1, \rho_2, \rho_3, t_1, t_2$), which are constrained to belong to the nonconvex uncertainty set described by the constraints (34) and (35).

Deterministic bifurcation analysis

Let $\mathscr{D}^c$ be the imaginary axis of the complex place. i.e.,

$$\mathscr{D}^c = \{\lambda \in \mathbb{C} : \lambda = \lambda_{\mathrm{re}} + j\lambda_{\mathrm{im}}, \quad \lambda_{\mathrm{re}}, \lambda_{\mathrm{im}} \in \mathbb{R}, \ \lambda_{\mathrm{re}} = 0\}.$$

For fixed width k of the intervals $[\rho_i^{\mathrm{o}} - k\Delta\rho_i \quad \rho_i^{\mathrm{o}} + k\Delta\rho_i]$, the deterministic bifurcation analysis problem can be formulated as a $\mathscr{D}$-stability analysis problem, or equivalently, in terms of problem (13), by assuming to know only the support of the uncertain parameters ρ_1, ρ_2, ρ_3. An upper bound $\overline{p}^\tau$ of $\overline{p}$ (i.e., solution of (13)) is computed by solving the relaxed SDP problem (24) for a relaxation order $\tau = 3$. Based on the considerations in Property 1, if $\overline{p}^\tau < 1$, then $J(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}})$ is guaranteed to have no eigenvalues on the imaginary axis for any $\rho_i \in [\rho_i^{\mathrm{o}} - k\Delta\rho_i \quad \rho_i^{\mathrm{o}} + k\Delta\rho_i]$. A bisection on the width k of the uncertainty intervals $[\rho_i^{\mathrm{o}} - k\Delta\rho_i \quad \rho_i^{\mathrm{o}} + k\Delta\rho_i]$ is then carried out to compute (a lower bound of) the maximum value of k such that $J(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}})$ is guaranteed not to have any eigenvalues on the imaginary axis for any $\rho_i \in [\rho_i^{\mathrm{o}} - k\Delta\rho_i \quad \rho_i^{\mathrm{o}} + k\Delta\rho_i]$. The obtained value of k is $k = 0.4620$ (similar to the result obtained in [22]) and the CPU time required to solve problem (24) for fixed k is, in average, 536 seconds. Since sufficient conditions on robust $\mathscr{D}$-stability are derived from $\overline{p}^\tau$, we can claim that the system is guaranteed to have no local bifurcation at the equilibrium point $(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}})$ for any ρ_i in the interval $[\rho_i^{\mathrm{o}} - k\Delta\rho_i \quad \rho_i^{\mathrm{o}} + k\Delta\rho_i]$, with $i = 1,2,3$ and $k = 0.4620$.

In this example, tightness of the computed solution can be verified analytically. In fact, the determinant of $J(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}})$ is $\alpha\frac{\rho_3}{\rho_1}(\rho_1 - \rho_3 - \rho_2\rho_3)$, which is equal to zero for $\rho_1 = 8.5412$, $\rho_2 = \rho_3 = 2.4650$. This values of ρ_1, ρ_2 and ρ_3 lie in the intervals $[\rho_i^{\mathrm{o}} - k\Delta\rho_i \quad \rho_i^{\mathrm{o}} + k\Delta\rho_i]$ for $k = 0.4650$.

Probabilistic bifurcation analysis

Let us now consider the case where the uncertain parameters ρ_i belong to the intervals $\rho_i \in [\rho_i^{\mathrm{o}} - \Delta\rho_i \quad \rho_i^{\mathrm{o}} + \Delta\rho_i]$, $i = 1,2,3$. The expected values $\mathbb{E}[\rho_i]$ of all the three uncertain parameters are equal to their nominal values ρ_i^{o}. Furthermore, we assume that an upper bound $\overline{\sigma}^2$ on the variance of the probability measure Pr_ρ describing the parameters ρ_i is known, i.e.,

$$\int_\Delta (\rho_i - \rho_i^{\mathrm{o}})^2 \, dP_\rho(\rho) \le \overline{\sigma}^2, \quad i = 1,2,3.$$

The solution $\overline{p}^\tau$ of the corresponding SDP problem (24) is computed for a relaxation order $\tau = 3$ and for different values of the (upper bound on the) variance $\overline{\sigma}^2$. Fig. 2 shows the computed upper probability $\overline{p}^\tau$ of the system to have a local bifurcation at the equilibrium point $(r_{1,\mathrm{eq}}, r_{2,\mathrm{eq}})$ for different values of $\overline{\sigma}^2$. It can be observed that, although for a width

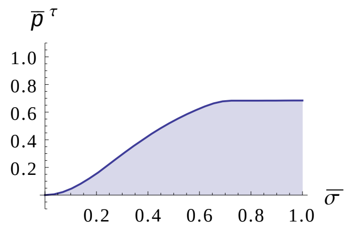

Fig. 2. Bifurcation analysis: maximum standard deviation $\overline{\sigma}$ of the probability measures $\Pr_{\rho}$ vs. upper probability $\overline{p}^{\tau}$ of having a local bifurcation.

$k=1$ of the uncertainty intervals the system is not guaranteed to have no local bifurcation, under the considered assumptions on the mean and the maximum variance, the probability that the system has a local bifurcation at the equilibrium point $(r_{1,\text{eq}}, r_{2,\text{eq}})$ is, in the worst-case scenario, smaller than 0.1 for $\overline{\sigma}^2$ smaller than 0.15^2. *In other words the system has not local bifurcation with probability at least* 0.9, considering an interval width $k=1$ (that is more than two times the one considered in the deterministic case ($k=0.462$)). *We can thus be much less conservative and at the same guaranteeing no bifurcation with "high probability".*

VI. Conclusions

In this paper, we have presented a unified framework for deterministic and probabilistic analysis of $\mathscr{D}$-stability of uncertain matrices. A generalized moment optimization problem is formulated and then relaxed through the Lasserre's hierarchy into a sequence of semidefinite programming problems of finite size. The relaxed problems provide lower bounds on the minimum probability of a family of matrices to be $\mathscr{D}$-stable. This is equivalent, in the deterministic realm, to derive sufficient conditions for robust $\mathscr{D}$-stability.

Stability regions $\mathscr{D}$ whose complement is described by polynomial constraints can be handled. This class of stability sets is quite vast and includes: (*i*) the open left half plane and the unit circle, which allows us to verify stability of continuous- and discrete-time LTI systems with parametric uncertainty; (*ii*) the imaginary axis, which allows us to compute an upper bound on the H_{∞}-norm of uncertain LTI systems; (*iii*) the semi-axis of positive real numbers, which allows us to verify positive definiteness of uncertain real symmetric matrices; (*iv*) the origin of the complex plane, which allows us to verify nonsingularity of uncertain matrices. Actually, the developed approach is widely applicable and it can be used to tackle many problems in systems and control theory.

Acknowledgment

The authors thank Prof. Nicola Guglielmi for the interesting discussions on $\mathscr{D}$-stability analysis and Prof. Johan Löfberg for his suggestions on the use of YALMIP.

References

[1] P. Walley, *Statistical Reasoning with Imprecise Probabilities*. New York: Chapman and Hall, 1991.

[2] A. Benavoli, M. Zaffalon, and E. Miranda, "Robust filtering through coherent lower previsions," *IEEE Trans. on Automatic Control*, vol. 56, no. 7, pp. 1567 –1581, 2011.

[3] A. Benavoli, "The generalized moment-based filter," *Automatic Control, IEEE Transactions on*, vol. 58, no. 10, pp. 2642–2647, 2013.

[4] A. Benavoli and D. Piga, "A probabilistic interpretation of set-membership filtering: application to polynomial systems through polytopic bounding," *Automatica*, vol. 70, pp. 158–172, 2016.

[5] J. B. Lasserre, "Global optimization with polynomials and the problem of moments," *SIAM J. on Optimization*, vol. 11, pp. 796–817, 2001.

[6] A. Nemirovskii, "Several NP-hard problems arising in robust stability analysis," *Mathematics of Control, Signals, and Systems*, vol. 6, no. 2, pp. 99–105, 1993.

[7] J. Rohn, "Systems of linear interval equations," *Linear algebra and its applications*, vol. 126, pp. 39–78, 1989.

[8] D. Hertz, "The extreme eigenvalues and stability of real symmetric interval matrices," *IEEE Trans. on Automatic Control*, vol. 37, no. 4, pp. 532–535, 1992.

[9] J. Rohn, "An algorithm for checking stability of symmetric interval matrices," *IEEE Trans. on Automatic Control*, vol. 41, no. 1, pp. 133–136, 1996.

[10] D. Henrion, D. Arzelier, and D. Peaucelle, "Positive polynomial matrices and improved LMI robustness conditions," *Automatica*, vol. 39, no. 8, pp. 1479–1485, 2003.

[11] T. Alamo, R. Tempo, D. R. Ramírez, and E. F. Camacho, "A new vertex result for robustness problems with interval matrix uncertainty," *Systems & Control Letters*, vol. 57, no. 6, pp. 474–481, 2008.

[12] V. Dzhafarov and T. Büyükköroğlu, "On the stability of a convex set of matrices," *Linear algebra and its applications*, vol. 414, no. 2, pp. 547–559, 2006.

[13] D. Peaucelle, D. Arzelier, O. Bachelier, and J. Bernussou, "A new robust d-stability condition for real convex polytopic uncertainty," *Systems & Control Letters*, vol. 40, no. 1, pp. 21–30, 2000.

[14] G. Chesi, "Establishing stability and instability of matrix hypercubes," *Systems & control letters*, vol. 54, no. 4, pp. 381–388, 2005.

[15] R. B. Barmish, *New Tools for Robustness of Linear Systems*. Macmillan, New York, NY, 1994.

[16] V. Blondel and J. N. Tsitsiklis, "A survey of computational complexity results in systems and control," *Automatica*, vol. 36, no. 9, pp. 1249–1274, 2000.

[17] R. Tempo, G. Calafiore, and F. Dabbene, *Randomized algorithms for analysis and control of uncertain systems: with applications*. Springer Science & Business Media, 2012.

[18] N. Guglielmi and C. Lubich, "Low-rank dynamics for computing extremal points of real pseudospectra," *SIAM Journal on Matrix Analysis and Applications*, vol. 34, no. 1, pp. 40–66, 2013.

[19] M. A. Freitag and A. Spence, "A new approach for calculating the real stability radius," *BIT Numerical Math.*, vol. 54, no. 2, pp. 381–400, 2014.

[20] I. R. Petersen and R. Tempo, "Robust control of uncertain systems: Classical results and recent developments," *Automatica*, vol. 50, no. 5, pp. 1315–1335, 2014.

[21] M. Rostami, "New algorithms for computing the real structured pseudospectral abscissa and the real stability radius of large and sparse matrices," *SIAM Journal on Scientific Computing*, vol. 37, no. 5, pp. 447–471, 2015.

[22] M. K. Kishida and R. D. Braatz, "On the analysis of the eigenvalues of uncertain matrices by μ and ν: Applications to bifurcation avoidance and convergence rates," *IEEE Trans. on Automatic Control*, vol. 61, no. 3, pp. 748–753, 2016.

[23] D. Henrion and J. B. Lasserre, "Inner approximations for polynomial matrix inequalities and robust stability regions," *IEEE Transactions on Automatic Control*, vol. 57, no. 6, pp. 1456–1467, 2012.

[24] R. Heß, D. Henrion, J. B. Lasserre, and T. S. Pham, "Semidefinite approximations of the polynomial abscissa," *SIAM Journal on Control and Optimization*, vol. 54, no. 3, pp. 1633–1656, 2016.

[25] D. Piga, "Computation of the Structured Singular Value via Moment LMI Relaxations," *IEEE Trans. on Automatic Control*, vol. 61, no. 2, pp. 520–525, 2016.

[26] D. Piga and A. Benavoli, "A unified framework for deterministic and probabilistic $\mathscr{D}$-stability analysis of uncertain polynomial matrices," *Technical report, IDSIA*, 2016. [Online]. Available: https://arxiv.org/abs/1604.02031

[27] J. Löfberg, "YALMIP: A toolbox for modeling and optimization in Matlab," in *IEEE International Symposium on Computer Aided Control Systems Design*, Taipei, Taiwan, 2004, pp. 284–289.

[28] Y. A. Kuznetsov, *Elements of applied bifurcation theory*. Springer Science & Business Media, 2013, vol. 112.