Computational load reduction in bounded error identification of Hammerstein systems

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***Abstract*—In this paper we present a procedure for the identification of Hammerstein systems from measurements affected by bounded noise. First, we show that computation of tight parameter bounds requires the solution to nonconvex optimization problems where the number of decision variables increases with the length of the experimental data sequence. Then, in order to reduce the computational burden of the identification problem, we propose a procedure to relax the formulated problem into a collection of polynomial optimization problems where the number of variables does not depend on the number of measurements. Advantages of the presented approach with respect to previously published results are discussed and highlighted by means of a simulation example.**

***Index Terms*—Hammerstein systems, Convex relaxation, Parameter bounds, Set-membership identification.**

I. Introduction

Nonlinear system identification has attracted the attention of many authors in the past decades. In spite of the rich literature, modeling and identification of nonlinear dynamic systems still remains a difficult task. One of the major challenge is the search for simple and flexible model structures which take into account the most relevant nonlinear phenomena encountered in practice. This problem has stimulated a number of contributions about the identification of block-structured nonlinear systems, which are obtained through interconnection of memoryless nonlinear functions and linear dynamic subsystems (see book [1] for an up-to-date collection of results and algorithms in this context). The configuration we are dealing with in this paper, commonly referred to as Hammerstein model, is shown in Fig. 1 and consists of a static nonlinear part $\mathcal{N}$ followed by a linear dynamic system $\mathcal{L}$. The identification of such a model relies solely on input-output measurements, while the inner signal x_t, i.e. the input of the linear block, is not assumed to be available. A number of algorithms can be found in the literature to address such a problem (see, e.g., [2]–[5]). In this paper we consider the identification of single-input single-output (SISO) Hammerstein models when the nonlinear block can be modeled by a linear combination of a finite and known number of nonlinear static functions, and the linear dynamic part is described by an output error model with bounded output measurement errors. As shown in [6], the Hammerstein system identification problem in the presence of bounded noise is NP-hard in the size of the experimental data sequence. In a previous paper by the authors [7], a two-stage identification procedure is presented, whose main limitation is that it requires two different experiments where two specific input signals have to be used, while in [8] a one-stage procedure without particular constraints on the input signal is presented, which requires the solution to polynomial optimization problems. The algorithm proposed in [8], however, suffers from computational burden because the number of decision variables grows with the length of the experimental data sequence. In this paper we present an alternative one-stage approach (referred to as semi-static relaxation) that significantly reduces the computational complexity of the algorithm proposed in [8]. The paper is organized as follows. Section II is devoted to the formulation of the identification problem, where we show that computation of tight parameter bounds requires the solution to nonconvex optimization problems where the number of decision variables increases with the number of measured data. A relaxation procedure to reduce the number of decision variables of such optimization problems is presented in Section III and a detailed analysis of its properties is reported in Section IV. A sufficient condition to check if the computed bounds are tight is provided in Section V. On the basis of such a condition, a systematic procedure to evaluate the degree of conservativeness of the computed bounds is also presented. In Section VI, the effectiveness of the proposed approach is shown by means of a simulation example. Proofs of the main results can be found in [9].

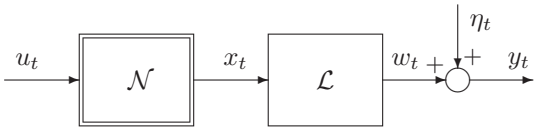

Fig. 1. Hammerstein system.

II. Problem setting

Consider the SISO discrete-time Hammerstein model depicted in Fig. 1. The nonlinear block $\mathcal{N}$ maps the input signal u_t into the unmeasurable inner variable x_t through the nonlinear function

$$x_t = \sum_{k=1}^{n_\gamma} \gamma_k \varphi_k(u_t), \quad t = 1, \ldots, N, \tag{1}$$

where $(\varphi_1,.....,\varphi_{n_\gamma})$ is a known basis of nonlinear functions and N is the length of data sequence. The linear dynamic part $\mathcal{L}$ is modeled by a discrete-time system transforming x_t into

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the noise-free output w_t according to equation

$$w_t = -\sum_{i=1}^{na} a_i w_{t-i} + \sum_{j=0}^{nb} b_j x_{t-j}. \quad (2)$$

Let y_t be the noise-corrupted output:

$$y_t = w_t + \eta_t. \quad (3)$$

Then, by combining eqs. (1), (2) and (3), the mapping between the input signal u_t and the noise-corrupted output signal y_t is given by:

$$y_t = -\sum_{i=1}^{na} a_i(y_{t-i} - \eta_{t-i}) + \sum_{j=0}^{nb}\sum_{k=1}^{n_\gamma} b_j\gamma_k\varphi_k(u_{t-j}) + \eta_t. \quad (4)$$

Measurement uncertainty η_t is known to range within given bounds $\Delta\eta_t$, i.e.,

$$\mid \eta_t \mid \leq \Delta\eta_t. \quad (5)$$

The unknown system parameters are collected into the vectors $\gamma = \left[\gamma_1, \gamma_2, \ldots, \gamma_{n_\gamma}\right]^{\mathrm{T}} \in \mathbb{R}^{n_\gamma}$ and $\theta = [a_1, \ldots, a_{na}, b_0, \ldots, b_{nb}]^{\mathrm{T}} \in \mathbb{R}^{n_\theta}$, with $n_\theta = na + nb + 1$. It must be pointed out that the parametrization of the structure in Fig. 1 is not unique. To get a unique parametrization we assume, without loss of generality, that the steady-state gain of the linear part is one, i.e.

$$1 + \sum_{i=1}^{na} a_i = \sum_{j=0}^{nb} b_j. \quad (6)$$

Then, the set $\mathcal{D}$ of parameters γ and θ and noise samples η_t consistent with observed data, assumed model structure and error bounds is described by (4), (5) and (6), i.e.

$$\begin{aligned}\mathcal{D} = \Big\{(\gamma,\theta,\eta) \in \mathbb{R}^{n_\gamma+n_\theta+N} : y_t = -\sum_{i=1}^{na} a_i(y_{t-i} - \eta_{t-i}) + \\ + \sum_{j=0}^{nb}\sum_{k=1}^{n_\gamma} b_j\gamma_k\varphi_k(u_{t-j}) + \eta_t, 1 + \sum_{i=1}^{na} a_i = \sum_{j=0}^{nb} b_j, \\ |\eta_r| \leq \Delta\eta_r, \;\; r = 1, \ldots, N; \;\; t = na+1, \ldots, N\Big\},\end{aligned} \quad (7)$$

with $\eta = [\eta_1, \ldots, \eta_N]^{\mathrm{T}}$. Therefore, for each $k = 1, \ldots, n_\gamma$ and $j = 1, \ldots, n_\theta$, tight bounds on parameters γ_k and θ_j can be computed by solving the constrained optimization problems

$$\underline{\gamma}_k = \min_{(\gamma,\theta,\eta)\in\mathcal{D}} \gamma_k, \;\; \overline{\gamma}_k = \max_{(\gamma,\theta,\eta)\in\mathcal{D}} \gamma_k, \quad (8a)$$

$$\underline{\theta}_j = \min_{(\gamma,\theta,\eta)\in\mathcal{D}} \theta_j, \;\; \overline{\theta}_j = \max_{(\gamma,\theta,\eta)\in\mathcal{D}} \theta_j. \quad (8b)$$

Thus, parameter uncertainty intervals on γ_k and θ_j are defined as $PUI_{\gamma_k} = \left[\underline{\gamma}_k;\, \overline{\gamma}_k\right]$ and $PUI_{\theta_j} = \left[\underline{\theta}_j;\, \overline{\theta}_j\right]$. Because of the product between the variable a_i and the noise η_{t-i}, as well as the product between the unknown parameters b_j and γ_k appearing in the equality constraints defining the feasible parameter set (FPS) $\mathcal{D}$ in eq. (7), problems (8) are nonconvex polynomial optimization problems. Therefore, standard nonlinear optimization tools (gradient method, Newton method, etc.) cannot be used to compute a solution to problems (8) since they can trap in local minima/maxima. As a consequence, the computed uncertainty intervals are underestimated and they are not guaranteed to contain the true system parameters. A possible solution to overcome such a problem is to relax identification problems (8) into convex problems. As discussed in [8], (8) are semialgebraic problems with inherent structured sparsity that can be relaxed by exploiting the LMI-relaxation techniques proposed in [10] and [11]. The method proposed in [8] leads to relaxed semidefinite programming (SDP) problems with a number of optimization variables which linearly increases with the number of measurements N and, because of high computational complexity, it can be applied only if N is, roughly, not greater than 1500. In this paper we present an alternative approach, referred to as semi-static relaxation, to relax (8) into polynomial problems where only the parameters γ and θ are considered as optimization variables, thus allowing numerical computation of parameter bounds for large values of N.

III. Semi-static relaxation

For the sake of clarity a general overview of the proposed method is first presented in Section III-A. Then, detailed technical results are given in Section III-B.

A. Overview of the method

The key steps of the semi-static approach are summarized in the following Algorithm.

Algorithm 1: Semi-static relaxation approach

A.1 Given an integer $n \in [1,\ N - na]$, for $z = 1,\ldots,N - na$, let us define the set $S_z^{(n)}$ as

$$\begin{aligned}S_z^{(n)} = \Big\{(\gamma,\theta,\eta) \in \mathbb{R}^{n_\gamma+n_\theta+N} : y_t = -\sum_{i=1}^{na} a_i(y_{t-i} - \eta_{t-i}) + \\ + \sum_{j=0}^{nb}\sum_{k=1}^{n_\gamma} b_j\gamma_k\varphi_k(u_{t-j}) + \eta_t, \; 1 + \sum_{i=1}^{na} a_i = \sum_{j=0}^{nb} b_j, \\ t = na+z, \ldots, \min\{na+z-1+n, N\}, \\ |\eta_r| \leq \Delta\eta_r, \;\; r = z, \ldots, \min\{na+z-1+n, N\}\Big\}.\end{aligned} \quad (9)$$

It is worth noticing that the sets $\mathcal{S}_z^{(n)}$ are described by at most n consecutive constraints defining $\mathcal{D}$ in (7). More precisely, $S_1^{(n)}$ is only defined by the constraints obtained from measurements at time $t = na+1$ up to time $t = na+n$, $\mathcal{S}_2^{(n)}$ is defined by the constraints obtained from measurements at time $t = na + 2$ up to time $t = na + n + 1$, and so on, up to $\mathcal{S}_{N-na}^{(n)}$ that is defined only by the constraint obtained from measurement at time N. The integer n is referred to as *dynamic horizon*.

A.2 **for** $z = 1 : N - na$

A.2.1 Denote the equality constraint $y_t = -\sum_{i=1}^{na} a_i(y_{t-i} - \eta_{t-i}) + \sum_{j=0}^{nb}\sum_{k=1}^{n_\gamma} b_j\gamma_k\varphi_k(u_{t-j}) + \eta_t$ appearing in the definition of $S_z^{(n)}$ in (9) as E_t.

A.2.2 **for** $t = na+z : \min\{na+z-1+n, N\}-1$

- Substitute equation E_t into equation E_{t+1} and denote as E_{t+1} the new obtained equality;
- Rewrite the set $S_z^{(n)}$ defined in (9) as

$$S_z^{(n)} = \big\{(\gamma,\theta,\eta)\in\mathbb{R}^{n_\gamma+n_\theta+N}\colon E_t,\ 1+\sum_{i=1}^{na}a_i=\sum_{j=0}^{nb}b_j,$$
$$t=na+z,\dots,\min\{na+z-1+n,N\},$$
$$|\eta_r|\le\Delta\eta_r,\ r=z,\dots,\min\{na+z-1+n,N\}\big\}.$$

end for

A.2.3 Construct an outer approximation $S_z^{ss(n)}$ of the set $S_z^{(n)}$ by assuming that all the occurrences of the same noise samples η_t appearing in the constraints defining $S_z^{(n)}$ at step A.2.2 are independent variables.

end for

A.3 $S_{N-na}^{ss(n)} \leftarrow S_{N-na}^{(n)}$.

A.4 Define an outer approximation $\mathcal{D}^{ss(n)}$ of $\mathcal{D}$ as $\mathcal{D}^{ss(n)} = \bigcap_{z=1}^{N-na} S_z^{ss(n)}$.

A.5 For all $k=1,\dots,n_\gamma$ and $j=1,\dots,n_\theta$, compute relaxed bounds on parameters γ_k and θ_j by solving optimization problems

$$\underline{\gamma}_k^{ss(n)} = \min_{\gamma,\theta\in\mathcal{D}^{ss(n)}}\gamma_k,\ \overline{\gamma}_k^{ss(n)} = \max_{\gamma,\theta\in\mathcal{D}^{ss(n)}}\gamma_k, \quad (10a)$$
$$\underline{\theta}_j^{ss(n)} = \min_{\gamma,\theta\in\mathcal{D}^{ss(n)}}\theta_j,\ \overline{\theta}_j^{ss(n)} = \max_{\gamma,\theta\in\mathcal{D}^{ss(n)}}\theta_j. \quad (10b)$$

■

Thanks to the structure of $\mathcal{D}^{ss(n)}$, the computation of parameter bounds in (10) will be formulated in terms of polynomial optimization problems with $n_\gamma+n_\theta$ decision variables, instead of the $n_\gamma+n_\theta+N$ variables involved in (8). It is worth remarking that, equality constraints defining the set $S_z^{(n)}$ are deterministically related through the uncertain noise samples η_t affecting successive output samples y_t. Because of that correlation, conservativeness is introduced at stage A.2.3 of Algorithm 1 in approximating $\mathcal{D}$ with $\mathcal{D}^{ss(n)}$. In order to reduce conservativeness, nested substitutions among the equality constraints defining the set $S_z^{(n)}$ are first made at stage A.2.2. In this way, correlation among constraints is partly preserved.

B. Technical results

Technical details of the semi-static relaxation method are here discussed. For a given integer $n\in[1,N-na]$, let us consider the alternative description $\mathcal{D}$ given by the intersection of the sets $\mathcal{S}_1^{(n)},\dots,\mathcal{S}_{N-na}^{(n)}$, i.e.

$$\mathcal{D} = \bigcap_{z=1}^{N-na}\mathcal{S}_z^{(n)}. \quad (11)$$

Let us rewrite the generic constraint $y_t = -\sum_{i=1}^{na}a_i(y_{t-i}-\eta_{t-i}) + \sum_{j=0}^{nb}\sum_{k=1}^{n_\gamma}b_j\gamma_k\varphi_k(u_{t-j})+\eta_t$ appearing in the definition of $\mathcal{S}_z^{(n)}$ in the matrix form $\underline{y}_t = F(\underline{y}_{t-1}-\underline{\eta}_{t-1}) + G\underline{\phi}_t + \underline{\eta}_t$, where

$$\underline{y}_t = [y_t, y_{t-1},\dots,y_{t-na+1}]^{\mathrm{T}},\quad \underline{\eta}_t = [\eta_t,\eta_{t-1},\dots,\eta_{t-na+1}]^{\mathrm{T}},$$
$$\underline{\phi}_t = \big[\varphi_1(u_t),\varphi_1(u_{t-1}),\dots,\varphi_1(u_{t-nb}),\varphi_2(u_t),\dots,\varphi_{n_\gamma}(u_{t-nb})\big]^{\mathrm{T}}$$

while matrices $F\in\mathbb{R}^{na,na}$ and $G\in\mathbb{R}^{na,nb\cdot n_\gamma}$ are defined as:

$$F = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{na-1} & -a_{na}\\ 1 & 0 & \dots & 0 & 0\\ 0 & 1 & \dots & 0 & 0\\ \vdots & . & \ddots & . & \vdots\\ 0 & 0 & \dots & 1 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} \gamma_1 b_0 & \dots & \gamma_1 b_{nb} & \gamma_2 b_0 & \dots & \gamma_{n_\gamma} b_{nb}\\ 0 & \dots & 0 & 0 & \dots & 0\\ \vdots & . & . & . & . & \vdots\\ 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Then, from nested substitution of equations defining $\mathcal{S}_z^{(n)}$ in (9), the set of constraints appearing in the definition of $\mathcal{S}_z^{(n)}$ can be alternatively written in terms of matrices F and G as

$$y_{na+z+s-1} = [1\ 0\dots\ 0]\Big[F^s\left(\underline{y}_{na+z-1}-\underline{\eta}_{na+z-1}\right) + \sum_{j=1}^{s}F^{s-j}G\underline{\phi}_{na+z+j-1} + \underline{\eta}_{na+z+s-1}\Big],$$
$$s=1,2,\dots,\min\{n,N-na-z+1\},$$

leading to the following alternative definition of $\mathcal{S}_z^{(n)}$:

$$\begin{aligned}\mathcal{S}_z^{(n)} = \Big\{&(\gamma,\theta,\eta)\in\mathbb{R}^{n_\gamma+n_\theta+N}:\\ &y_{na+z+s-1} = \sum_{i=1}^{na}a_i^{(s)}\left(y_{na+z-i}-\eta_{na+z-i}\right) +\\ &+\sum_{j=1}^{s}\left\langle\underline{g}_j^{(s)},\underline{\phi}_{na+z+j-1}\right\rangle + \eta_{na+z+s-1},\\ &1+\sum_{i=1}^{na}a_i = \sum_{j=0}^{nb}b_j,\ \ |\eta_r|\le\Delta\eta_r,\\ &s=1,\dots,\min\{n,N-na-z+1\},\\ &r=z,\dots,\min\{na+n+z-1,N\}\Big\},\end{aligned} \quad (12)$$

where coefficients $a_i^{(s)}$ as well as components of the vector $\underline{g}_j^{(s)}$, with $s=1,2,\dots,\min\{n,N-na-z+1\}$, are polynomial functions of unknown parameters γ and θ. In particular $a_i^{(s)} = [F^s](1,i)$ and $\underline{g}_j^{(s)} = [F^{s-j}G](1,:)$, with $[R](1,i)$ denoting the entry of the first row and the $i-th$ column of a generic matrix R, $[R](1,:)$ is the first row of R, while $\left\langle\underline{g}_j^{(s)},\underline{\phi}_{na+z+j-1}\right\rangle$ denotes the inner product between the vectors $\underline{g}_j^{(s)}$ and $\underline{\phi}_{na+z+j-1}$.

Result 1: **Construction of an outer bound of $\mathcal{S}_z^{(n)}$**

Let us define the set $\mathcal{S}_z^{ss(n)}$ as

$$\begin{aligned}\mathcal{S}_z^{ss(n)} = \big\{&(\gamma,\theta,\eta)\in\mathbb{R}^{n_\gamma+n_\theta+N}:\\ &y_{na+z+s-1}-\sum_{i=1}^{na}a_i^{(s)}y_{na+z-i}-\sum_{j=1}^{s}\left\langle\underline{g}_j^{(s)},\underline{\phi}_{na+z+j-1}\right\rangle\le\\ &\le\Delta\eta_{na+z+s-1}+\sum_{i=1}^{na}a_i^{(s)}\mathrm{sign}(a_i^{(s)})\Delta\eta_{na+z-i};\\ &y_{na+z+s-1}-\sum_{i=1}^{na}a_i^{(s)}y_{na+z-i}-\sum_{j=1}^{s}\left\langle\underline{g}_j^{(s)},\underline{\phi}_{na+z+j-1}\right\rangle\ge\\ &\ge-\Delta\eta_{na+z+s-1}-\sum_{i=1}^{na}a_i^{(s)}\mathrm{sign}(a_i^{(s)})\Delta\eta_{na+z-i};\\ &1+\sum_{i=1}^{na}a_i = \sum_{j=0}^{nb}b_j,\ \ s=1,\dots,\min\{n,N-na-z+1\}\big\}.\end{aligned}$$

Then, for all $z = 1,\ldots,N-na$, the set $\mathcal{S}_z^{ss(n)}$ is an outer approximation of $\mathcal{S}_z^{(n)}$, i.e. $\mathcal{S}_z^{(n)} \subseteq \mathcal{S}_z^{ss(n)}$. ■

Result 2: **Construction of an outer bound of $\mathcal{D}$**

The set $\mathcal{D}^{ss(n)}$ defined as $\mathcal{D}^{ss(n)} = \bigcap_{z=1}^{N-na} \mathcal{S}_z^{ss(n)}$ is an outer approximation of the exact FPS $\mathcal{D}$. ■

On the basis of Result 2, for a given dynamic horizon n and for all $k = 1,\ldots,n_\gamma$ and $j = 1,\ldots,n_\theta$, relaxed parameter uncertainty intervals on γ and θ can be defined as $PUI_{\gamma_k}^{ss(n)} = \left[\underline{\gamma}_k^{ss(n)};\, \overline{\gamma}_k^{ss(n)}\right]$ and $PUI_{\theta_j}^{ss(n)} = \left[\underline{\theta}_j^{ss(n)};\, \overline{\theta}_j^{ss(n)}\right]$, with $\underline{\gamma}_k^{ss(n)}$, $\overline{\gamma}_k^{ss(n)}$, $\underline{\theta}_j^{ss(n)}$ and $\overline{\theta}_j^{ss(n)}$ given in (10).

Remark 1: Only the unknown Hammerstein system parameters γ and θ are optimization variables for problems (10); on the contrary in the original identification problems (8) also the noise samples η are treated as variables. ■

Evaluation of intervals $PUI_{\gamma_k}^{ss(n)}$ and $PUI_{\theta_j}^{ss(n)}$ requires the solution to optimization problems (10) over the nonconvex feasible region $\mathcal{D}^{ss(n)}$. In the following we describe how to numerically solve such problems by exploiting the particular structure of $\mathcal{D}^{ss(n)}$. In order to analyze the topological features of the set $\mathcal{D}^{ss(n)}$, we first introduce the following notation. Let Γ be the set of all vectors with $n \cdot na$ components, each one equal to either $+1$ or -1. This means that $\Gamma = \{\alpha_1, \alpha_2, \ldots, \alpha_l, \ldots, \alpha_L\}$, where $L = 2^{n \cdot na}$ and α_l is a vector with $n \cdot na$ components, each one equal to either $+1$ or -1 and such that: $\alpha_l \neq \alpha_i$ if $l \neq i$ for all $l, i = 1,\ldots,L$. Let $q^{(n)} \in \mathbb{R}^{na \cdot n}$ be the collection of polynomials $a_i^{(s)}$ for $i = 1,\ldots,na$ and $s = 1,\ldots,n$, i.e. $q^{(n)} = \left[a_1^{(1)},\ldots,a_{na}^{(1)},a_1^{(2)},\ldots,a_{na}^{(2)},\ldots,a_{na}^{(n)}\right]^{\mathrm{T}}$. For any $\alpha_l \in \Gamma$, let us define the set $\mathcal{O}^{(n)}(\alpha_l) \subset \mathbb{R}^{n_\gamma+n_\theta}$ as $\mathcal{O}^{(n)}(\alpha_l) = \left\{(\gamma,\theta) \in \mathbb{R}^{n_\gamma+n_\theta} : \alpha_{lj} q_j^{(n)}(\theta) \geq 0,\ j = 1,\ldots,n \cdot na\right\}$, where α_{lj} and $q_j^{(n)}$ are the j-th element of the vectors α_l and $q^{(n)}$, respectively. Topological features of $\mathcal{D}^{ss(n)}$ are now highlighted by the following result.

Result 3: The set $\mathcal{D}^{ss(n)}$ is the union of at most L sets $\mathcal{D}_l^{ss(n)}$ in $\mathbb{R}^{n_\gamma+n_\theta+N}$, that is $\mathcal{D}^{ss(n)} = \bigcup_{l=1}^{L} \mathcal{D}_l^{ss(n)}$, where $\mathcal{D}_l^{ss(n)} = \mathcal{D}^{ss(n)} \cap \mathcal{O}^{(n)}(\alpha_l)$. Furthermore, the set $\mathcal{D}_l^{ss(n)}$, if not empty, is a semialgebraic region in $\mathbb{R}^{n_\gamma+n_\theta+N}$. ■

Statement of Result 3 follows from the fact that in each set $\mathcal{O}^{(n)}(\alpha_l)$ the sign of each component of the vector $q^{(n)}$, i.e. the sign of polynomials $a_i^{(s)}$, is imposed.

Thanks to the structure of $\mathcal{D}^{ss(n)}$ highlighted by Result 3, problems (10) can be decomposed into a collection of polynomial optimization problems, that is

$$\underline{\gamma}_k^{ss(n)} = \min_{l=1,\ldots,L} \underline{\gamma}_{kl}^{ss(n)},\ \overline{\gamma}_k^{ss(n)} = \max_{l=1,\ldots,L} \overline{\gamma}_{kl}^{ss(n)}, \tag{13a}$$

$$\underline{\theta}_j^{ss(n)} = \min_{l=1,\ldots,L} \underline{\theta}_{jl}^{ss(n)},\ \overline{\theta}_j^{ss(n)} = \max_{l=1,\ldots,L} \overline{\theta}_{jl}^{ss(n)}, \tag{13b}$$

where

$$\underline{\gamma}_{kl}^{ss(n)} = \min_{\gamma,\theta \in \mathcal{D}_l^{ss(n)}} \gamma_k,\ \overline{\gamma}_{kl}^{ss(n)} = \max_{\gamma,\theta \in \mathcal{D}_l^{ss(n)}} \gamma_k, \tag{14a}$$

$$\underline{\theta}_{jl}^{ss(n)} = \min_{\gamma,\theta \in \mathcal{D}_l^{ss(n)}} \theta_j,\ \overline{\theta}_{jl}^{ss(n)} = \max_{\gamma,\theta \in \mathcal{D}_l^{ss(n)}} \theta_j. \tag{14b}$$

Since the optimization variables involved in (14) are only unknown parameters γ and θ, evaluation of relaxed solutions to such polynomial problems by means of LMI-relaxation techniques [12], [13] and [14] is computationally tractable. In particular, for a given relaxation order $\delta \geq \underline{\delta} = \lceil \frac{n+1}{2} \rceil$, application of the theory-of-moment relaxation in [12] to problems (14) leads to the following SDP problems:

$$\underline{\gamma}_{kl}^{ss(n,\delta)} = \min_{p \in \mathcal{D}_l^{ss(n,\delta)}} f_k(p),\quad \overline{\gamma}_{kl}^{ss(n,\delta)} = \max_{p \in \mathcal{D}_l^{ss(n,\delta)}} f_k(p), \tag{15}$$

$$\underline{\theta}_{jl}^{ss(n,\delta)} = \min_{p \in \mathcal{D}_l^{ss(n,\delta)}} h_j(p),\quad \overline{\theta}_{jl}^{ss(n,\delta)} = \max_{p \in \mathcal{D}_l^{ss(n,\delta)}} h_j(p), \tag{16}$$

defining: (i) lower bounds $\underline{\gamma}_{kl}^{ss(n,\delta)}$ and $\underline{\theta}_{jl}^{ss(n,\delta)}$ of $\underline{\gamma}_{kl}^{ss(n)}$ and $\underline{\theta}_{jl}^{ss(n)}$ and (ii) upper bounds $\overline{\gamma}_{kl}^{ss(n,\delta)}$ and $\overline{\theta}_{jl}^{ss(n,\delta)}$ of $\overline{\gamma}_{kl}^{ss(n)}$ and $\overline{\theta}_{jl}^{ss(n)}$, respectively. In (15) and (16), p is the sequence of moments up to order 2δ associated with variables γ and θ whose dimension is $\begin{pmatrix} n_\gamma + n_\theta + 2\delta \\ 2\delta \end{pmatrix}$. The objective functions $f_k(p)$, $h_j(p)$ in (15)-(16) are linear in the moments p, and the feasible region $\mathcal{D}_l^{ss(n,\delta)}$ is a convex set described by linear matrix inequalities obtained by relaxing nonconvex problems (14) through the theory-of-moment approach. The reader is referred to [12] for technical details on the relaxation of polynomial problems through the theory of moments.

Remark 2: The computation of $\underline{\gamma}_k^{ss(n,\delta)}$, $\overline{\gamma}_k^{ss(n,\delta)}$, $\underline{\theta}_j^{ss(n,\delta)}$ or $\overline{\theta}_j^{ss(n,\delta)}$ requires the solution to at most $L = 2^{n \cdot na}$ SDP problems (15) or (16). In fact, problems (14) have to be solved over at most L feasible regions $\mathcal{D}_l^{ss(n)}$. Indeed, a deterministic correlation between the signs of some elements of the vector $q^{(n)}$ could be used to reduce the number of feasible regions $\mathcal{D}_l^{ss(n)}$ where optimization problems (14) have to be solved. For instance, some components of $q^{(n)}$ could take the same value. Thus, the regions $\mathcal{D}_l^{ss(n)}$, where the signs of such components are different, are empty sets and problems (14) can not be solved in such regions. Besides, let us suppose that a_1, a_2 and $a_1 a_2$ are components of $q^{(n)}$. Thus, the sign of the product $a_1 a_2$ is uniquely determined by the sign of a_1 and a_2. An alternative way to reduce the number of optimization problems to be solved, is to firstly compute parameter bounds for dynamic horizon $n = 1$ in order to detect which sets $\mathcal{D}_l^{ss(n)}$ are empty for values of $n > 1$. ■

For a given dynamic horizon $n \geq 1$ and a relaxation order $\delta \geq \underline{\delta}$, let us define the parameter uncertainty intervals $PUI_{\gamma_k}^{ss(n,\delta)} = \left[\underline{\gamma}_k^{ss(n,\delta)};\, \overline{\gamma}_k^{ss(n,\delta)}\right]$ and $PUI_{\theta_j}^{ss(n,\delta)} = \left[\underline{\theta}_j^{ss(n,\delta)};\, \overline{\theta}_j^{ss(n,\delta)}\right]$, where

$$\underline{\gamma}_k^{ss(n,\delta)} = \min_{l=1,\ldots,L} \underline{\gamma}_{kl}^{ss(n,\delta)},\ \overline{\gamma}_k^{ss(n,\delta)} = \max_{l=1,\ldots,L} \overline{\gamma}_{kl}^{ss(n,\delta)}, \tag{17}$$

$$\underline{\theta}_j^{ss(n,\delta)} = \min_{l=1,\ldots,L} \underline{\theta}_{jl}^{ss(n,\delta)},\ \overline{\theta}_j^{ss(n,\delta)} = \max_{l=1,\ldots,L} \overline{\theta}_{jl}^{ss(n,\delta)}. \tag{18}$$

IV. Properties of obtained parameter uncertainty intervals

In this section we show that the computed parameter uncertainty intervals $PUI_{\gamma_k}^{ss(n,\delta)}$ enjoy a number of properties. Similar results hold for $PUI_{\theta_j}^{ss(n,\delta)}$.

Property 1: **Guaranteed relaxed uncertainty intervals.** For every dynamic horizon $n \in [1, N-na]$ and for any relaxation order $\delta \geq \underline{\delta} = \left\lceil \frac{n+1}{2} \right\rceil$ the interval $PUI_{\gamma_k}^{ss(n,\delta)}$ is guaranteed to contain the true nonlinear block parameter γ_k^o to be estimated, i.e. $\gamma_k^o \in PUI_{\gamma_k}^{ss(n,\delta)}$ for all $k = 1, \ldots, n_\gamma$.

Property 2: **Monotone convergence to parameter uncertainty intervals $PUI_{\gamma_k}^{ss(n)}$.** For every dynamic horizon $n \in [1, N-na]$ and relaxation order $\delta \geq \underline{\delta} = \left\lceil \frac{n+1}{2} \right\rceil$ the parameter uncertainty interval $PUI_{\gamma_k}^{ss(n,\delta)}$ becomes tighter as the relaxation order δ increases, that is $PUI_{\gamma_k}^{ss(n,\delta+1)} \subseteq PUI_{\gamma_k}^{ss(n,\delta)}$. Furthermore, the computed interval $PUI_{\gamma_k}^{ss(n,\delta)}$ converges to $PUI_{\gamma_k}^{ss(n)}$ as the relaxation order δ goes to infinity, i.e. $\lim_{\delta\to\infty} \underline{\gamma}_k^{ss(n,\delta)} = \underline{\gamma}_k^{ss(n)}$ and $\lim_{\delta\to\infty} \overline{\gamma}_k^{ss(n,\delta)} = \overline{\gamma}_k^{ss(n)}$. ■

Since $\underline{\gamma}_k^{ss(n,\delta)}$ and $\overline{\gamma}_k^{ss(n,\delta)}$ are relaxed with respect to tight bounds $\underline{\gamma}_k$ and $\overline{\gamma}_k$, statement of Property 1 follows. Property 2 follows from monotone convergence properties of the theory-of-moment relaxation [12].

Property 3: **Increasing accuracy in uncertainty intervals evaluation.** For every dynamic horizon $n \in [1, N-na-1]$ and relaxation order $\delta \geq \underline{\delta} = \left\lceil \frac{n+1}{2} \right\rceil$, the parameter uncertainty interval $PUI_{\gamma_k}^{ss(n,\delta)}$ becomes tighter as the dynamic horizon n increases, i.e. $PUI_{\gamma_k}^{ss(n+1,\delta)} \subseteq PUI_{\gamma_k}^{ss(n,\delta)}$. ■

It is worth remarking that, on the basis of Remark 2 and Property 3, the tradeoff between computational complexity and accuracy of the semi-static relaxation approach can be balanced by tuning the dynamic horizon n.

V. On the accuracy of relaxed uncertainty intervals

In this section we present a systematic procedure to measure the level of accuracy of the relaxed uncertainty intervals $PUI_{\gamma_k}^{ss(n,\delta)}$ and $PUI_{\theta_j}^{ss(n,\delta)}$ with respect to tight intervals PUI_{γ_k} and PUI_{θ_j}. Let $\underline{l}$ be the index which solves the minimization problem in (17), i.e $\underline{l}$ is such that $\underline{\gamma}_{k\underline{l}}^{ss(n,\delta)} = \underline{\gamma}_k^{ss(n,\delta)}$. Let $\underline{p}^{(n,\gamma_k)}$ be the optimizer of the minimization problem in (15) for $l = \underline{l}$, and let $\underline{\gamma}^{(n,k)} \in \mathbb{R}^{n_\gamma}$ and $\underline{\theta}^{(n,k)} \in \mathbb{R}^{n_\theta}$ be the collection of first-order moments of the sequence $\underline{p}^{(n,\gamma_k)}$ associated with parameters γ and θ, respectively. Consider the following optimization problem:

$$
\begin{aligned}
\underline{\rho}_{\underline{\gamma}}^{(n,k)} = \min_{\rho,\eta}\ & \rho \\
s.t.\ \ & y_t = -\sum_{i=1}^{na} a_i (y_{t-i} - \eta_{t-i}) + \sum_{j=0}^{nb}\sum_{k=1}^{n_\gamma} b_j \gamma_k \varphi_k(u_{t-j}) + \eta_t, \\
& 1 + \sum_{i=1}^{na} a_i = \sum_{j=0}^{nb} b_j, \ \ \gamma = \underline{\gamma}^{(n,k)}, \ \theta = \underline{\theta}^{(n,k)}, \ \ |\eta_r| \leq \rho \Delta\eta_r \\
& t = na+1, \ldots, N; \ \ r = 1, \ldots, N.
\end{aligned}
\tag{19}
$$

$\underline{\rho}_{\underline{\gamma}}^{(n,k)}$ is referred to as the coefficient of tightness of the relaxed parameter bound $\underline{\gamma}_k^{\delta}$. Note that, since system parameters γ and θ are enforced to be equal to $\underline{\gamma}^{(n,k)}$ and $\underline{\theta}^{(n,k)}$, respectively, (19) is a linear optimization problem. Problem (19) provides the minimum value of the error bound $\rho \cdot \Delta\eta_t$ such that there exists a sequence of noise samples $\eta_1, \ldots, \eta_N$, with $|\eta_t| \leq \rho \cdot \Delta\eta_t$, such that system parameters $\gamma = \underline{\gamma}^{(n,k)}$ and $\theta = \underline{\theta}^{(n,k)}$ are consistent with measured data and error bound $\rho \cdot \Delta\eta_t$. To put it another way, problem (19) provides the minimum value of the error bound such that $\gamma = \underline{\gamma}^{(n,k)}$ and $\theta = \underline{\theta}^{(n,k)}$ belong to the feasible parameter set $\mathcal{D}$. On the basis of the above considerations, the result below follows.

Result 4: If $\underline{\rho}_{\underline{\gamma}}^{(n,k)} \leq 1$, then the relaxed bound $\underline{\gamma}_k^{ss(n,\delta)}$ is tight, i.e., $\underline{\gamma}_k^{ss(n,\delta)} = \underline{\gamma}_k$. ■

Result 4 provides a sufficient condition to check if the relaxed bound $\underline{\gamma}_k^{\delta}$ computed for a given value of δ is tight. On the other hand, $\underline{\rho}_{\underline{\gamma}}^{(n,k)}$ provides an index of the degree of conservativeness of the computed bound. In fact, when $\underline{\rho}_{\underline{\gamma}}^{(n,k)} > 1$, $\underline{\gamma}_k^{\delta}$ is certainly not smaller than the minimum value of γ_k belonging to the feasible parameter set $\mathcal{D}$ corresponding to noise bound $\underline{\rho}_{\underline{\gamma}}^{(n,k)} \cdot \Delta\eta_t$. Through similar considerations, we can compute the coefficients of tightness $\rho_{\overline{\gamma}}^{(n,k)}$, $\rho_{\underline{\theta}}^{(n,j)}$ and $\rho_{\overline{\theta}}^{(n,j)}$ of relaxed bounds $\overline{\gamma}_k^{ss(n,\delta)}$, $\underline{\theta}_j^{ss(n,\delta)}$ and $\overline{\theta}_j^{ss(n,\delta)}$, respectively.

VI. A numerical example

Let us consider a Hammerstein system where the nonlinear block is modeled by $x_t = 0.3u_t + 0.4u_t^2 - 0.6u_t^3$ and the linear part is a second-order model with true parameter vector $\theta^o = [a_1^o\ a_2^o\ b_1^o\ b_2^o]^{\mathrm{T}} = [0.65\ 0.73\ 1.41\ 0.97]^{\mathrm{T}}$. The output data sequence is corrupted by random additive noise η_t uniformly distributed in $[-\Delta\eta_t,\ +\Delta\eta_t]$ and the chosen error bounds $\Delta\eta_t$ are such that the signal to noise ratio $SNR_w = 10\log\left\{\sum_{t=1}^{N} w_t^2 \Big/ \sum_{t=1}^{N} \eta_t^2\right\}$ is equal to 25 db. The length of the data sequence is $N = 1500$. Bounds on the parameters are evaluated by solving (15) and (16) for a relaxation order $\delta = 2$. The Matlab package Gloptipoly [15] has been used to convert the original identification problems (14) into the corresponding LMI relaxed problems (15) and (16), which are numerically solved by the SDP solver SeDuMi. The uncertainty intervals are evaluated for two different values of dynamic horizon n, i.e. $n = 2$ and $n = 3$. In order to compare the semistatic approach with the sparse LMI-based technique presented in [8], uncertainty intervals are also computed with the method in [8]. Results on the evaluation of nonlinear and linear block parameters are reported in Table I and II, respectively, which show the obtained parameter bounds, the central estimates $\gamma_k^{c(n,\delta)} = \left(\overline{\gamma}_k^{ss(n,\delta)} + \underline{\gamma}_k^{ss(n,\delta)}\right)/2$ and $\theta_j^{c(n,\delta)} = \left(\overline{\theta}_j^{ss(n,\delta)} + \underline{\theta}_j^{ss(n,\delta)}\right)/2$, parameter uncertainties $\Delta\gamma_k^{(n,\delta)} = \left(\overline{\gamma}_k^{ss(n,\delta)} - \underline{\gamma}_k^{ss(n,\delta)}\right)/2$ and $\Delta\theta_j^{(n,\delta)} = \left(\overline{\theta}_j^{ss(n,\delta)} - \underline{\theta}_j^{ss(n,\delta)}\right)/2$, coefficients of tightness $\rho_{\underline{\gamma}}^{(n,k)}$, $\rho_{\overline{\gamma}}^{(n,k)}$, $\rho_{\underline{\theta}}^{(n,j)}$ and $\rho_{\overline{\theta}}^{(n,j)}$ and the total CPU time taken to compute both the lower and the upper bounds on each parameter.

The reported results show that as the dynamic horizon n grows, the accuracy in the evaluation of parameter uncertainty intervals increases, according to Property 3. It is worth remarking that, for dynamic horizon $n = 3$, the coefficients of tightness are close to 1. This means that the relaxed bounds

TABLE I

NONLINEAR BLOCK. PARAMETER BOUNDS ($\underline{\gamma}_k^{ss(n,\delta)}$, $\overline{\gamma}_k^{ss(n,\delta)}$), CENTRAL ESTIMATES $\gamma_k^{c(n,\delta)}$, PARAMETER UNCERTAINTIES $\Delta\gamma_k^{(n,\delta)}$, COEFFICIENTS OF TIGHTNESS ($\rho_{\underline{\gamma}}^{(n,k)}$,$\rho_{\overline{\gamma}}^{(n,k)}$) AND CPU TIME TAKEN TO COMPUTE PARAMETER BOUNDS.

Relaxation approach	Parameter	True value	$\underline{\gamma}_k^{ss(n,\delta)}$	$\gamma_k^{c(n,\delta)}$	$\overline{\gamma}_k^{ss(n,\delta)}$	$\Delta\gamma_k^{(n,\delta)}$	$\rho_{\underline{\gamma}}^{(n,k)}$	$\rho_{\overline{\gamma}}^{(n,k)}$	CPU Time [s]
semistatic	γ_1	0.300	0.247	0.294	0.341	0.047	1.49	1.42	757
$(n=2)$	γ_2	0.400	0.328	0.395	0.461	0.066	1.14	1.25	779
	γ_3	-0.600	-0.849	-0.585	-0.321	0.264	1.83	1.92	741
semistatic	γ_1	0.300	0.274	0.299	0.323	0.024	1.04	1.05	987
$(n=3)$	γ_2	0.400	0.358	0.404	0.450	0.046	1.03	1.13	1024
	γ_3	-0.600	-0.741	-0.623	-0.506	0.118	1.19	1.21	1017
sparse	γ_1	0.300	0.279	0.309	0.339	0.030	1.00	1.08	19472
LMI-based	γ_2	0.400	0.329	0.383	0.437	0.054	1.13	1.02	20638
[8]	γ_3	-0.600	-0.705	-0.615	-0.523	0.091	1.00	1.05	20382

TABLE II

LINEAR BLOCK. PARAMETER BOUNDS ($\underline{\theta}_j^{ss(n,\delta)}$, $\overline{\theta}_j^{ss(n,\delta)}$), CENTRAL ESTIMATES $\theta_j^{c(n,\delta)}$, PARAMETER UNCERTAINTIES $\Delta\theta_j^{(n,\delta)}$, COEFFICIENTS OF TIGHTNESS ($\rho_{\underline{\theta}}^{(n,j)}$,$\rho_{\underline{\theta}}^{(n,j)}$) AND CPU TIME TAKEN TO COMPUTE PARAMETER BOUNDS.

Relaxation approach	Parameter	True value	$\underline{\theta}_j^{ss(n,\delta)}$	$\theta_j^{c(n,\delta)}$	$\overline{\theta}_j^{ss(n,\delta)}$	$\Delta\theta_j^{(n,\delta)}$	$\rho_{\underline{\theta}}^{(n,j)}$	$\rho_{\overline{\theta}}^{(n,j)}$	CPU Time [s]
semistatic	a_1	0.650	0.560	0.642	0.724	0.082	2.89	1.82	774
$(n=2)$	a_2	0.730	0.575	0.752	0.928	0.176	2.83	2.15	725
	b_1	1.410	1.294	1.421	1.548	0.127	1.72	1.88	763
	b_2	0.970	0.860	0.975	1.090	0.115	2.25	2.01	794
semistatic	a_1	0.650	0.623	0.667	0.710	0.043	1.05	1.14	1012
$(n=3)$	a_2	0.730	0.718	0.758	0.798	0.040	1.04	1.32	993
	b_1	1.410	1.328	1.417	1.506	0.090	1.21	1.14	984
	b_2	0.970	0.913	0.977	1.041	0.064	1.13	1.12	1028
sparse	a_1	0.650	0.625	0.657	0.688	0.032	1.02	1.01	20845
LMI-based	a_2	0.730	0.721	0.731	0.740	0.010	1.01	1.00	19952
[8]	b_1	1.410	1.332	1.413	1.494	0.081	1.03	1.08	20181
	b_2	0.970	0.932	0.980	1.027	0.048	1.01	1.02	20831

$\underline{\gamma}_k^{ss(n,\delta)}$, $\overline{\gamma}_k^{ss(n,\delta)}$, $\underline{\theta}_j^{ss(n,\delta)}$ and $\overline{\theta}_j^{ss(n,\delta)}$ are close to the tight bounds $\underline{\gamma}_k$, $\overline{\gamma}_k$, $\underline{\theta}_j$ and $\overline{\theta}_j$. Furthermore, for $n = 3$, the semistatic method provides parameter uncertainties $\Delta\gamma_k^{(n,\delta)}$ and $\Delta\theta_j^{(n,\delta)}$ similar to the ones evaluated through the LMI-based approach proposed in [8], in a time which is about 20 times smaller. Furthermore, parameter uncertainties on γ_1 and γ_2 computed with semistatic approach are smaller than the ones evaluated through the method discussed in [8].

VII. CONCLUSIONS

A procedure for the evaluation of bounds on the parameters of Hammerstein systems in the presence of bounded errors is presented. Parameter bound evaluation is formulated as a collection of constrained polynomial optimization problems, with a number of variables that increases with the number of measurements. In order to reduce the computational complexity of the identification problems, an outer bound of the feasible parameter set is sought, which is the union of a finite number of semialgebraic sets over the parameter space. Thus parameter bounds can be evaluated by solving suitable polynomial optimization problems involving a smaller number of variables, i.e. only the unknown parameters of the system. Then, LMI relaxation techniques are used to approximate global optima in order to compute guaranteed parameter uncertainty intervals. The main advantage of the presented procedure with respect to previous set-membership techniques is that a design parameter, referred in the paper to as dynamic horizon, can be tuned in order to balance the tradeoff between accuracy and computational complexity.

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